FREE VIBRATION STUDIES ON NON-HOMOGENEOUS CIRCULAR AND ANNULAR PLATES

A THESIS

Submitted in partial fulfilment of the requirements for the award of the degree

of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

by

SEEMA SHARMA

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
ROORKEE-247 667 (INDIA)

JULY, 2006

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ROORKEE

CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled FREE VIBRATION STUDIES ON NON-HOMOGENEOUS CIRCULAR AND ANNULAR PLATES in partial fulfillment of the requirements for the award of the Degree of Doctor of Philosophy and submitted in the Department of Mathematics of the Indian Institute of Technology Roorkee, Roorkee is an authentic record of my own work carried out during the period from [illegible] to [illegible] under the supervision of Dr. Roshan Lal, Professor and Dr. U.S. Gupta, Emeritus Professor, Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee.

[illegible]

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other Institute.

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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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The Ph.D. Viva-Voce Examination of Ms. [illegible], Research Scholar, has been held on [illegible].

Signature of Supervisor(s)

Signature of External Examiner

INDIAN INSTITUTE OF TECHNOLOGY, ROORKEE
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The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

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(Seema Sharma)

This is to certify that the above statement made by the candidate is correct to the best of our knowledge.

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Signature of the External Examiner

To

All the Helping Forces

of

NATURE

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Seema Sharma

(SEEMA SHARMA)

RESEARCH PAPERS PRESENTED/PUBLISHED IN THE NATIONAL/INTERNATIONAL CONFERENCES AND JOURNAL

1. Lal, R., Gupta, U.S. and Sharma Seema, "Axisymmetric vibrations of non-homogeneous polar orthotropic annular plates of variable thickness resting on an elastic foundation", Proc. Conf. of Indian Society of Mechanical Engineers held at I.I.T. Roorkee, Dec. 30-31, pp. MD-074, 2003.

2. Lal, R., Gupta, U.S. and Sharma Seema, "Axisymmetric vibrations of non-homogeneous annular plate of quadratically varying thickness", Proc. Int. Conf. on Advances in Applied Mathematics(ICAAM-05) held at Gulbarga University. Gulbarga. Feb. 24-26, pp. 167-181, 2005.

3. Gupta, U.S., Lal, R. and Sharma Seema, "Axisymmetric vibration of polar orthotropic annular plate of variable thickness resting on Pasternak foundation". Conf. on IMS held at I.I.T. Roorkee, Dec. 27-29, 2005.

4. Gupta, U.S., Lal, R. and Sharma Seema, "Vibration analysis of non-homogeneous circular plate of non-linear thickness variation by differential quadrature method". (in press), Journal of Sound & Vibration, 2006, U.K.

5. Gupta, U.S., Lal, R. and Sharma Seema, "Vibration of non-homogeneous circular Mindlin plates with variable thickness", (in press), Journal of Sound & Vibration, 2006, U.K.

6. Gupta, U.S., Lal, R. and Sharma Seema, "Thermal effect on axisymmetric vibrations of non-uniform polar orthotropic circular plates with elastically restrained edge", accepted for presentation in Int. Conf. (ICCMS-06) to be held in December, 2006 at I.I.T. Guwahati.

CONTENTS

ABSTRACT

The object of the work presented in this thesis is an attempt to study the free vibrational behaviour of isotropic/polar-orthotropic non-homogeneous circular and annular plates with various complicating effects such as thickness variation, elastic foundation, thermal gradient, shear deformation, rotatory inertia and elastically restrained edge. Very little work dealing with non-homogeneous plates is available in the literature. The model proposed herein to account for non-homogeneity of plate material is such that most of the earlier proposed models can be regarded as particular cases. The thesis consists of **nine** chapters. Chapter I presents an up-to-date survey of literature on vibration of plates with various complicating effects. The remaining work from chapters II to IX is divided into **two parts**, A and B. **Part A**(chapters II to V), deals with isotropic plates, while **part B**(chapters VI to IX), deals with polar-orthotropic plates. Extensive numerical results for the frequencies and mode shapes for various values of plate parameters have been given in each chapter, which would be of interest to design engineers.

The chapter-wise summary is given as follows:

<u>PART A</u>

Chapter II deals with free axisymmetric vibrations of isotropic non-homogeneous annular plate of quadratically varying thickness on the basis of classical plate theory. The non-homogeneity of the plate material is assumed to arise due to the variation of Young's modulus and density which are assumed to vary exponentially in the radial direction. The numerical solution of the governing differential equation derived by using Hamilton's energy principle is obtained by
D Q M
differential quadrature method (DQM), which provides highly accurate results with minimal

computational efforts. First three natural frequencies have been computed for different values of various plate parameters such as non-homogeneity, density, taper and also radii ratio for three different combinations of boundary conditions. Mode shapes for the first three modes of vibration are computed for specified plates. The results for linear as well as parabolic thickness variations have been obtained as special cases. Comparison of results with those available in the literature has been presented.

In Chapter III, an analysis for the free axisymmetric vibrations of isotropic non-homogeneous circular plate of variable profile has been presented on the basis of classical plate theory. Assuming the exponential variation for non-homogeneity of the plate material and quadratic variation for thickness as in chapter II, the differential quadrature method has been used to obtain the frequency equations for three different edge conditions. The effect of non-homogeneity, density and taper parameters and that of edge conditions on natural frequencies have been investigated for the first three modes of vibration. Normalized transverse displacements have been presented for a specified plate for all the three edge conditions. Special cases for linear as well as parabolic thickness variations have been deduced. A comparison of results with those available in literature by other methods has been presented. A comparative study for evaluation of frequencies for specified plates with respect to different choices of grid points has also been carried out.

Chapter IV deals with free axisymmetric vibrations of isotropic non-homogeneous, moderately thick annular plates of quadratically varying thickness. The analysis is based on a set of coupled differential equations with variable coefficients derived by an extension of Mindlin's plate theory. As a closed form solution of these equations is not feasible, an approximate

solution has been obtained using DQM. For three different combinations of edge conditions. frequency equations have been solved in respect of different values of thickness. non-homogeneity, density and taper parameters and also radii ratio to obtain the first three natural frequencies. Transverse displacements are presented for specified plates for first three modes of vibration. The results have been compared with those available in literature by other methods. A comparison of frequencies with the corresponding values obtained by classical plate theory has also been presented.

In chapter V, the effect of transverse shear and rotatory inertia on flexural vibrations of isotropic non-homogeneous circular plates of variable thickness has been studied. The governing differential equations derived in chapter IV have been extended for circular plates of quadratically varying thickness in radial direction. Chebyshev collocation technique has been employed for their numerical solution with exponential variation for non-homogeneity of the plate material. The effect of various plate parameters such as thickness, non-homogeneity, density and taper on the frequencies for three different edge conditions has been analyzed. Mode shapes for specified plates have been presented for the first three modes of vibration. Comparison of frequencies for isotropic homogeneous Mindlin circular plate of uniform thickness obtained by other methods has been presented. A comparison of frequencies with those obtained by classical plate theory has also been made.

PART B

Chapter VI deals with the analysis of free axisymmetric vibrations of non-homogeneous polar orthotropic annular plates of non-uniform thickness on the basis of classical theory of plates. Assuming the quadratic variation for thickness and exponential variation for non-homogenity

solution has been obtained using DQM. For three different combinations of edge conditions. frequency equations have been solved in respect of different values of thickness. non-homogeneity, density and taper parameters and also radii ratio to obtain the first three natural frequencies. Transverse displacements are presented for specified plates for first three modes of vibration. The results have been compared with those available in literature by other methods. A comparison of frequencies with the corresponding values obtained by classical plate theory has also been presented.

In chapter V, the effect of transverse shear and rotatory inertia on flexural vibrations of isotropic non-homogeneous circular plates of variable thickness has been studied. The governing differential equations derived in chapter IV have been extended for circular plates of quadratically varying thickness in radial direction. Chebyshev collocation technique has been employed for their numerical solution with exponential variation for non-homogeneity of the plate material. The effect of various plate parameters such as thickness, non-homogeneity, density and taper on the frequencies for three different edge conditions has been analyzed. Mode shapes for specified plates have been presented for the first three modes of vibration. Comparison of frequencies for isotropic homogeneous Mindlin circular plate of uniform thickness obtained by other methods has been presented. A comparison of frequencies with those obtained by classical plate theory has also been made.

PART B

Chapter VI deals with the analysis of free axisymmetric vibrations of non-homogeneous polar orthotropic annular plates of non-uniform thickness on the basis of classical theory of plates. Assuming the quadratic variation for thickness and exponential variation for non-homogeneity

of plate material, an approximate solution of the governing differential equation of motion of such plates has been obtained by new version of DQM. First three natural frequencies have been computed for different values of radii ratio and various plate parameters, such as rigidity, non-homogeneity, density and taper parameters for three different combinations of boundary conditions. Normalized transverse displacements for the first three modes of vibration have been computed for specified plates. A comparison of minimum number of grid points to obtain the results with four digit exactitude by DQM and new version of DQM has been made. Also a comparison of results with those available in literature has been presented.

Chapter VII analyses the effect of Winkler type elastic foundation on free axisymmetric vibrations of polar orthotropic non-homogeneous annular plates of variable thickness on the basis of classical plate theory. The solution of the equations of motion for the plates of exponentially varying thickness has been obtained by employing the Chebyshev collocation technique. The effect of foundation together with orthotropy on the natural frequencies has been investigated for different values of radii ratio and various plate parameters such as taper, non-homogeneity and density for three different combinations of edge conditions. Mode shapes have been presented for a specified plate for the first three modes of vibration. A comparison of results with those available in literature has also been presented.

In chapter VIII, the effect of two-parameter elastic foundation (Pasternak) has been investigated on free axisymmetric vibrations of polar orthotropic non-homogeneous annular plates of variable profile on the basis of classical plate theory. An approximate solution of the governing differential equation for such plates has been obtained by Chebyshev collocation technique for inner edge clamped and outer edge clamped or simply supported. First three natural frequencies

for linearly as well as parabolically varying thickness have been computed for various values of non-homogeneity, density, taper, rigidity, foundation parameters and radii ratio. Normalized transverse displacements for the first three modes of vibration for specified plate have been plotted. A comparison of results with those available in the literature has also been presented.

Chapter IX investigates the effect of non-homogeneity caused by a constant thermal gradient on the free axisymmetric vibrations of polar orthotropic circular plates of quadratically varying thickness with elastically restrained edge on the basis of classical plate theory. An approximate solution of the problem is obtained by Ritz method, which employs polynomial coordinate functions as the basis functions. Mode shapes have been computed for a specified plate and the effect of orthotropy together with thermal gradient has been studied on the natural frequencies for various values of taper and flexibility parameters for the first three modes of vibration. The cases of classical boundary conditions i.e. clamped, simply supported and free edge conditions have been deduced as special cases.

CHAPTER I

INTRODUCTION

The study of vibration has acquired great importance due to its applications in various fields of engineering and technology ranging from household goods, such as washing machines, grinders and juicers, to aerospace industries. Vibration is a universal phenomenon since all bodies possessing mass and elasticity are capable of vibration. Atoms, molecules, nuclei and even the minutest particle in nature vibrate. There are low frequency vibrations of the lungs and heart, high frequency vibrations of the ear drum and also the vibrations induced by body motions. Depending upon their characteristics, vibrations may be desirable and undesirable. Musical instruments and machines would be of no interest without vibration. Vibration of telephone receivers, centrifugal separators, turbines, teleprinters and that of musical instruments is desirable, while that of machines and bridges is undesirable. Vibrations are also undesirable, when measurements with precision instruments, such as an electron microscope, are required. Vibrations are also beneficial for many purposes, such as atomic clocks that are based on atomic vibrations, ultrasonic instrumentation used in eye and other types of surgery. Vibrations may cause excessive noise due to wear of machine parts leading to reduction in performance. The knowledge of the theory of vibrations not only helps in preventing undesirable vibrations, but also helps in increasing the efficiency and life span of the machines. When a structure is excited with a frequency corresponding to one of its natural frequencies, then resonance occurs, resulting in excessive deflection and failure. Thus, the knowledge of natural frequencies of structural components is essential for design engineers to avoid resonance.

The study of vibration of plates has attracted scientists and engineers since the work of Chladni in 1787, who presented various modes of vibration for completely free square plates(Leissa[1973]). Later in 1811, Sophie Germain developed the theory for vibrating isotropic plates, which is known as classical plate theory. He derived the governing differential equation for such plates, but the boundary conditions were erroneous. The correct boundary conditions were given by G.R. Kirchoff(1850a, 1850b). This classical plate theory, also known as small deflection plate theory, is attributed to Kirchoff and Love(1850a, 1850b, 1944) and is based upon the following assumptions :

1. Thickness is small as compared to other dimensions.
2. The normal stresses in the direction transverse to the middle surface are taken to be negligibly small.
3. The middle surface of the plate remains unstretched during deformation.
4. The linear elements normal to the undeformed middle surface remain linear and normal to the deformed middle surface.

Several books have significantly contributed to the development of this subject. The Mathematical Theory of Elasticity by Love[1944] is one of the oldest and best books. Subsequently, many books have appeared dealing with bending and vibration of plates, notable ones amongst them being Volterra and Zachmanoglu[1965], Szilard[1974], Panc[1975], Gorman[1982], Timoshenko and Woinowsky-Krieger[1984], Shames and Dym[1985] and Rao[2004]. Based upon classical plate theory, a lot of research work dealing with vibration of plates of various geometries such as circular, annular, rectangular and triangular etc. can be found in the literature. Out of these shapes, circular and annular have been studied extensively.

Most of the work deals with plates of uniform thickness. An excellent survey of work up to 1965 has been given by Leissa in his monograph[1969]. Itao and Crandall[1979] have presented an interesting study giving as many as 701 lowest frequencies of uniform circular plates with free edge. In his bibliography, Naruoka[1981] has compiled 12717 research papers on the theory of plates published all over the world. Weisensel[1989] has given an elegant account of natural frequencies for homogeneous isotropic circular/annular plates of uniform thickness in two tables based upon 55 research papers. The study of vibrational behaviour of isotropic homogeneous plates has been continuing till today. Recently, Liew and Yang[1999, 2000] obtained full vibration spectrum of natural frequencies and mode shapes of circular and annular plates respectively, using three-dimensional analysis. Liu et al.[2001] studied axisymmetric vibration frequencies of some typical circular and annular plates by satisfying stress boundary conditions. Chakraverty et al.[2001] analysed free vibration of annular elliptic plates by using boundary characteristic orthogonal polynomials in Rayleigh-Ritz method. Rossi and Laura[2002] studied transverse vibrations of thin circular plates with mixed boundary conditions. Rossi[2002] presented an analysis of transverse vibrations of a circular annular plate with free inner edge. Zhou et al.[2003] analysed three-dimensional vibrational behaviour of circular and annular plates using Chebyshev-Ritz method. Zagrai and Donskoy[2005] obtained natural frequencies and modal parameters of uniform circular plates with elastic edge support. Lower natural frequencies of circular plates with mixed boundary conditions have been determined and analysed by Bauer and Eidel[2006].

There has been an increasing interest in the study of vibrational behaviour of circular and annular plates of variable thickness due to their increasing use in aerospace industry, electronic and optical equipments and missile technology. A considerable number of papers is available

for various types of thickness variation and different boundary conditions. Conway[1957, 1958] and Conway[1964] analysed flexural vibrations of discs of linearly varying thickness. Barakat and Bauman[1968] have considered parabolic thickness variation in the study of axisymmetric vibration of thin circular plates. Harris[1968], Kirkhope and Wilson[1972], Soni and Amba-Rao[1975], Chen[1976] and Luisoni et al.[1977] studied the axisymmetric as well as asymmetric vibrations of non-uniform isotropic circular plates. Gelos et al.[1981] studied the vibrations of circular plates with variable profile. Ficcadenti et al.[1986] performed numerical experiments on vibrating circular plates of non-uniform thickness and variable rotational constraint along the edge. Avalos et al.[1987] and Sanzi et al.[1989] presented an analysis of vibration of circular plates with stepped thickness over a concentric region. Selmane and Lakis[1999] studied natural frequencies of transverse vibrations of non-uniform circular and annular plates. Singh and Saxena[1995] and Bambill and Laura[1996] studied axisymmetric vibrations of circular plates with double linear thickness variations. Singh and Saxena[1996a, 1996b] obtained natural frequencies for transverse vibrations of circular plates with quadratic and exponential thickness variations. Chen[1997] analysed the axisymmetric vibrations of circular and annular plates of arbitrarily varying thickness. Singh and Hassan[1998] studied the transverse vibrations of circular plates with arbitrarily varying thickness. Recently, Duan et al.[2005] have given generalised hypergeometric function solutions for transverse vibration of a class of thin annular plates with thickness varying monotonically in radial direction.

As the plates used in various applications may have appreciable thickness, the classical plate theory over-predicts the natural frequencies because of overestimation of stiffness of the plate. The theory, however, does not account for the effect of transverse shear deformation and rotatory inertia. Research on incorporating shear deformation effects has resulted in many

theories which range from Reissner[1945], Hencky[1947], Uflyand[1948] to Mindlin[1951] and finally to Deresiewicz and Mindlin[1955], who introduced both the shear deformation and rotatory inertia effects in the study of vibrational behaviour of isotropic circular plates of uniform thickness. A literature survey by Liew et al.[1995] presents an excellent review of earlier research work/investigations on thick plate vibrations.

In the last few decades, various linear and higher order theories dealing with plate type structures have been developed and are given in Levinson[1980], Reddy and Phan[1985], Senthilnathan[1989] and Soedel[2004]. These theories have been widely used in most engineering applications. An extensive review of work dealing with complicating effects, such as thickness variation and axial force etc. has been given by Leissa in his monograph[1969] and a series of review articles[1977, 1978, 1981, 1987]. Later work is not found in the form of review articles, although quite a large number of papers on plate vibrations is available in the literature. Notable contributions are listed in references : Verma [1987], Kapania and Raciti[1989], Reddy [1990], Chakraverty [1992], Jain[1993], Saxena[1996], Bert and Malik[1996a, 1996b, 1996c], Chakraverty et al.[1999], Ansari[2000], Xiang[2002], Gupta and Ansari[2002], Kang[2003] and Sheikh et al.[2003]. Recently, Cheung and Zhou[2003] carried out an analysis of the free vibration of rectangular Mindlin plate with variable thickness. An exact solution of buckling and vibration of stepped rectangular Mindlin plate has been given by Xiang and Wei[2004]. Malekzadeh et al.[2004] presented free vibration analysis of Mindlin plates. Wang et al.[2004] presented mode shapes and modal stress resultants for freely vibrating circular plates. Free vibration analysis of moderately thick plates of variable thickness with elastically restrained edge has been given by Malekzadeh and Shahpari[2005]. Xiang and

Zhang[2005] gave an analysis of the vibrational behaviour of circular Mindlin plates with stepwise thickness variations.

In many practical situations, particularly in aerospace industry, missile technology and microelectronics, certain parts have to operate under high temperature environment causing non-homogeneity of the material i.e. elastic constants of the material become functions of space variables. Further, structural components are non-homogeneous either by design or due to imperfections in the underlying material. Material properties, in such elastic bodies, vary in a continuous manner. Ply-wood, timber and fibre-reinforced plastics etc. form an important class of non-homogeneous materials which are used in engineering applications. Very few models, representing the behaviour of non-homogeneity, have been proposed in the literature. The earliest model was proposed by Bose[1967], where Young's modulus and density are supposed to vary with radius vector. Biswas[1969], in his model, considered exponential variations for torsional rigidity and the material density, while Rao et al.[1974] have taken linear variations for both the Young's modulus and the density. In a series of papers, Tomar et al.[1982a, 1982b, 1983, 1984] have analysed the vibrational behaviour of non-homogeneous isotropic plates of variable thickness of different geometries on the basis of classical plate theory. The non-homogeneity of the plate material is assumed to arise due to the variation of Young's modulus and density along one direction. In the aforesaid models, the Poisson's ratio has been assumed to remain constant.

All the above investigations have been carried out assuming the plate material to be elastically isotropic. On account of the desirability of light weight, high strength, corrosion resistance and high temperature performance requirements in modern technology such as diaphragms used in

pressure capsules, aircraft fuselages, circular and annular plates stiffened with circumferential ribs and modern composite plates, to mention a few, there has been a phenomenal increase in the development of anisotropic materials during the last two decades of the last century. Many engineering structures made of composite materials may be modelled analytically as orthotropic(a special case of anisotropy). Composite materials find their use especially in aerospace technology, ocean engineering and applied sciences. Thus, the vibrational behaviour of fibre-reinforced materials and their increasing use have led to the study of vibration of orthotropic plates. Hearmon[1946] was the first known researcher in the field of rectangular orthotropy. Later on, Lekhnitskii[1968] extended the classical plate theory to anisotropic plate dealing with bending, stability and vibration. A considerable amount of work, dealing with vibration of polar orthotropic circular and annular plates, has been carried out by a number of researchers and the notable ones are reported in references (Ramaiah and Vijaikumar[1973], Ginesu et al.[1979], Avalos et al.[1982], Laura et al.[1982], Gorman[1982], Bell and Kirkhope[1983], Narita[1984], Reddy[1984], Narita et al.[1986], Ram Kumar et al.[1987], Gunaratnam and Bhattacharya[1989], Kim and Dickinson[1990], Bercin[1996], Davi and Milazzo[2003] and Lal and Sharma[2003, 2004a], Kang et al.[2005]).

The consideration of thickness variation together with orthotropy in the design of structural components, not only ensures reduction in size and weight, but also meets the desirability of economy and high strength. Keeping this in view, Laura et al.[1982] studied the effect of linear thickness variation on the vibration of polar orthotropic circular plates. Lal and Gupta[1982] analysed the axisymmetric vibrations of polar orthotropic annular plates of linearly varying thickness using Chebyshev collocation technique. Gorman[1983a, 1983b] used the finite element technique to obtain natural frequencies of circular and annular plates of variable

thickness under different conditions. Gupta and Lal[1985] analysed the effect of linear thickness variation on the axisymmetric vibrations of polar orthotropic Mindlin annular plates. Sankarnarayanan et al.[1985] studied the axisymmetric vibrations of layered annular plates with linear variation in thickness. Using Ritz method, Gupta and Ansari[1998b] studied the asymmetric vibration and elastic stability of polar orthotropic circular plates of linearly varying profile. In two papers, Gupta and Ansari[1998a, 2003] analysed the effect of linear and parabolic thickness variations on free vibrations of polar orthotropic circular plates with elastically restrained edge. Recently, Lal and Sharma[2004b] studied the vibration of non-homogeneous polar orthotropic annular plate of exponentially varying thickness. Elishakoff and Pentaras[2006] studied simply supported polar orthotropic inhomogeneous circular plates.

In normal practice, the actual boundary condition on a periphery, often tends to be partway between the classical conditions and may correspond more closely to some form of elastic restraint. The analysis of vibration of plates with elastically restrained edge is an important problem in aeronautical and naval structural engineering. In aircraft structures, the individual plates are connected to other plates or stiffened at their boundaries and thus, have elastic restraint at their edges. Many researchers, like Laura et al.[1976, 1981], have drawn attention to vibration of plates with elastic restraints and presented the fundamental frequencies of circular plates elastically restrained against rotation. Narita and Leissa[1980, 1981] analysed simply supported and free circular plates having elastic constraints. Leissa and Narita[1981] gave free vibration analysis of circular plates having elastic constraints and added mass, distributed along the edge segments. Irie et al.[1983] studied the free vibrations of circular plates with elastic restraint along the radial segment. Azimi[1988] studied the free vibrations of isotropic plates restrained elastically on their boundary by receptance method and also reviewed earlier work

on elastically restrained circular plates. Bambill et al.[1996] obtained the first mode approximation of mass and compliance of circular plates elastically restrained against rotation. Comparatively, little amount of work has been done dealing with vibration of polar orthotropic circular and annular plates with elastically restrained edge conditions. Notable contribution up to 2002, on vibration of polar orthotropic circular/annular plates of uniform and variable thickness with elastically restrained edge, are reported in references : Kim and Dickinson[1990], Gupta et al.[1991], Chung et al.[1993], Lawrence and Yin[1996], Gupta and Ansari[1998a, 1998b, 2002]. More recently, Singh and Jain[2003, 2004] studied asymmetric transverse vibrations of polar orthotropic annular plates of non-uniform thickness with elastically restrained edges. In another paper, Singh and Jain[2005] presented an analysis of free asymmetric vibration of polar orthotropic annular sector plate with a linearly varying thickness when circular edges are elastically restrained. In a latest study, Gupta et al.[2006] presented a study on asymmetric buckling and vibration of polar orthotropic circular plates of linearly varying thickness resting on Winkler foundation with elastically restrained edge.

Plates supported by elastic foundations have achieved great importance in foundation engineering such as floor slabs of multi-storey buildings, foundation of deep wells and storage tanks, pavement slabs of roads and airfields etc.[Szilard 1974, p.136]. A number of research workers has analysed the effect of elastic foundation on natural frequencies of plates. In this context, Hetenyi[1946, 1966] and Vlasov[1966] investigated the effect of elastic foundation on the dynamic behaviour of beams and plates. Various models, approximating the supporting medium(i.e. foundation), such as Vlasov, Pasternak and Winkler, have been proposed in the literature and an excellent account of these models, has been given by Kerr[1964]. Bhattacharya[1977] investigated the effect of Vlasov foundation on the natural frequencies of

triangular plates. Gupta and Lal[1979] and Gupta et al.[1985] analysed the effect of Winkler foundation and orthotropy on the frequencies of annular plates of linearly varying thickness using quintic splines interpolation technique. Later, Gupta et al.[1990] used Ritz method in the study of the effect of Winkler foundation on vibration of polar orthotropic circular plates of variable thickness when the edge is elastically restrained. Laura and Gutierrez[1991] studied the effect of Winkler type foundation on the vibration of a circular plate of linearly varying thickness. Gupta et al.[1994] analysed the effect of Winkler foundation on axisymmetric vibration of polar orthotropic circular plates. Liew et al.[1996] investigated the effect of Winkler type foundation on natural frequencies of Mindlin's rectangular plates. Ayvaz et al.[1998] applied a modified Vlasov model for the earthquake analysis of plates resting on elastic foundation. Gupta and Ansari[1999, 2002] and Gupta et al.[2006] studied the effect of elastic foundation (Winkler type) on asymmetric vibrations of polar orthotropic plates of variable thickness using Ritz method.

Clearly, most of the studies have been devoted to Winkler model of foundation, which is assumed to be replaced by a series of unconnected, closely spaced vertical elastic springs. Main disadvantage of Winkler's model is the discontinuity in the displacement. To avoid this disadvantage, a more realistic model was proposed by Pasternak. This model provides a close approximation to foundation reaction as it takes into account, not only its transverse reaction, but also shear interaction between spring elements, which is achieved by connecting the ends of the springs to the plate with incompressible vertical elements. Keeping this in view, a number of papers has appeared in the literature dealing with Pasternak type elastic foundation and is given in references (Nassar[1981], Kamal and Durvasula[1983], Dumir[1987], Paliwal[2003], Smaill[1991], Xiang et al.[1994], Han and Liew[1997], Wang et al.[1997], Shen[1997],

Omurtag and Kadioglu[1998], Shen et al.[2001], Wang et al.[2001], Filipich and Rosales[2002], Teo and Liew[2002], Malekzadeh and Karami[2004], Güler[2004], Zhou et al.[2006]), to mention a few. Among these, most of the work is concerned with bending of beams and plates of different geometries. The above up-to-date survey clearly shows that very little work has been done on the vibration of polar orthotropic circular /annular plates resting on Pasternak foundation.

Plate type structural components find wide applications in aerospace structures. In addition to mechanical loads, they are subjected to thermal load, which is caused by aerodynamic heating. Such thermal environments also arise in many engineering applications such as radiant burners, heat exchangers and artillery barrels etc. The extent of heating depends upon particular situation. The increase in temperature causes change in the material properties and thus, the homogeneous character of the material breaks down, giving rise to non-homogeneity in the material i.e. mechanical properties of the material become functions of space variables (Hoff[1958]). This necessitates the study of vibration analysis in the presence of thermal disturbances. Due to the increasing use of modern materials in the design of structural elements, there is an obvious need to study their vibrational behaviour in the presence of thermal gradient. Most of the engineering materials are found to have a linear relationship between the modulus of elasticity and temperature (Nowacki[1962], Fauconneau and Marangoni[1970]). A number of studies, dealing with the effect of thermal gradient on vibration of isotropic/orthotropic rectangular/circular/annular/elliptic plates of uniform/non-uniform thickness, has been reported in references: Irie and Yamada[1978], Ganesan and Dhotarad[1979], Laura and Gutierrez[1980], Gupta[1984], Tomar and Gupta[1983, 1984a, 1984b, 1985a, 1985b], Gorman[1983c, 1985a, 1985b], Ghosh[1985], Kar and Sujata[1988,

Dhotarad[1979], Laura and Gutierrez[1980], Gupta[1984], Tomar and Gupta[1983, 1984a, 1984b, 1985a. 1985b], Gorman[1983c. 1985a. 1985b]. Ghosh[1985], Kar and Sujata[1988. 1990], Rao et al.[1996]. During the survey of the literature, the author has not come across papers, dealing with linear vibrations of plates in the presence of thermal gradient, except that of Li and Zhou[2001] and Arafat et al.[2004], in which natural frequencies of heated annular and circular plates have been presented.

In addition to the above references, the following literature has also been consulted: Conway[1959]. Sternberg and Chakravorty[1959]. Thorkildsen and Hoppmann[1959], Erdogan[1985], Clastornik et al.[1986]. Tsai[1987], Joshi[1995]. Liu and Chen[1995], Masad[1996]. Gutierrez et al.[1996], Saha et al.[1997], Wang and Wang[2005], Hou et al.[2005] and Sharma[2005].

The above survey of literature reveals the fact that almost no work has been done dealing with vibrations of non-homogeneous plates of variable thickness during the last two decades with the exception of Sharma[2005], which presents free vibration studies on non-homogeneous annular plates. A major part is devoted to plates of either uniform or linear or exponential thickness variation using Chebyshev collocation technique. Thus. it is evident that no work has been done to study the effect of quadratic thickness variation, Pasternak foundation, thermal gradient and elastically restrained edge on the natural frequencies of non-homogeneous isotropic and orthotropic circular and annular plates.

Present Investigation

The object of the work presented in this thesis is an attempt to study the free vibrational behaviour of isotropic/polar-orthotropic non-homogeneous circular and annular plates with various complicating effects, such as thickness variation, shear deformation, rotatory inertia, elastic foundation (Winkler and Pasternak), thermal gradient and elastically restrained edge. The model considered to account for non-homogeneity of plate material is the same as taken by Sharma[2005]. Hamilton's energy principle has been used to derive the governing differential equation of motion for different plate problems, which has been solved by using four different numerical techniques, namely differential quadrature method, Chebyshev collocation technique, new version of differential quadrature method and Ritz method. Convergence studies have been carried out for all the numerical techniques employed. In case of DQM technique, an interesting study has been presented for evaluation of frequency parameter for a specified plate by taking equally spaced and three unequally spaced grid points i.e. zeros of shifted Legendre polynomials, zeros of shifted Chebyshev polynomials and grid points taken by Liew et al. [1997]. Extensive numerical results for frequencies and mode shapes have been obtained for various values of plate parameters for different boundary conditions. The results would be of great interest to design engineers in obtaining desired frequency by the proper choice of one or more plate parameters. The thesis consists of nine chapters. Chapter I presents an up-to-date survey of literature on vibration of plates with various complicating effects. Chapters II to V, deal with isotropic plates, while chapters VI to IX, deal with polar-orthotropic plates.

Chapters II and III deal with the axisymmetric vibrations of annular and circular plates, respectively, on the basis of classical plate theory, while Mindlin's plate theory has been used

in chapters IV and V for respective geometries. In all these chapters, the variation in thickness of the plate has been taken as quadratic. Differential quadrature method (DQM) has been used in chapters II-IV and Chebyshev collocation technique has been used in chapter V.

In chapter VI, new version of DQM has been used to study the axisymmetric vibrations of polar orthotropic annular plates of quadratically varying thickness. Chebyshev collocation technique has been used in chapters VII and VIII to study the effect of Winkler foundation and Pasternak foundation respectively, on the axisymmetric vibrations of annular plates. The thickness variation has been taken to be exponential in chapter VII and general in chapter VIII. In chapter IX, Ritz method has been used to analyse the effect of thermal gradient on the axisymmetric vibrations of circular plates of quadratically varying thickness with elastically restrained edge. In all the above chapters i.e. from VI-IX, the analysis has been presented on the basis of classical plate theory.

CHAPTER II

VIBRATIONS OF NON-HOMOGENEOUS ANNULAR PLATES OF QUADRATIC THICKNESS

1. INTRODUCTION

The study of vibrations of annular plates of variable thickness is of great interest in connection with various engineering applications such as aeronautical, civil, marine and mechanical engineering. In many practical situations, as in aerospace and missile technology, the structural components have to operate under elevated temperatures which cause non-homogeneity in the plate material i.e. elastic constants of the material become functions of the space variables. In an up-to-date survey of literature, the author has come across various models to account for non-homogeneity of plate material proposed by researchers dealing with vibration. Bose[1967] analyzed the vibrations of thin non-homogeneous circular plates with a central hole assuming the variation in Young's modulus and density in radial direction as $E = E_0 r$ and $\rho = \rho_0 r$, where E_0 and ρ_0 are constants. Biswas[1969], considered a non-homogeneous material for which rigidity $\mu = \mu_0 e^{-\mu_1 z}$ and density $\rho = \rho_0 e^{-\mu_1 z}$, where μ_0 and ρ_0 are constants and both μ and ρ were assumed to vary exponentially. Rao et al.[1974], dealing with vibration of non-homogeneous isotropic thin plates, have assumed linear variations for Young's modulus and density given by $E = E_0(1+\alpha x)$ and $\rho = \rho_0(1+\beta x)$. In a series of papers, Tomar et al.[1982a, 1982b, 1983, 1984] have assumed exponential variations i.e. $E = E_0\, e^{\mu x}$ and $\rho = \rho_0\, e^{\mu x}$ in the study of vibrational behaviour of non-homogeneous isotropic plates. The Poisson's ratio has been assumed to remain constant. The assumption of variation, in which parameter μ is the same for Young's modulus as well as density, does not seem to have any justification. In the

present study, a more general model has been proposed and used in which the Young's modulus and density have been assumed to vary exponentially in radial direction in a distinct manner.

Various numerical techniques such as Frobenius method (Tomar et al.[1982a]), finite-difference method (Singh et al.[1996a]), simple polynomial approximation method(Gupta et al.[1990], Avalos et al.[1982]), Galerkin's method (Ratko[2005]), Rayleigh-Ritz method (Gutierrez et al.[1996], Singh and Saxena[1996], Gupta and Ansari[2003]), characteristic orthogonal polynomial method (Singh and Chakraverty[1994]), quintic splines method (Lal et al.[1997]), finite element method (Chen and Ren[1998], Liu and Lee[2000]) and Chebyshev collocation method (Gupta et al.[1994], Lal et al.[2001]), etc. have been employed to study the vibrational characteristics of plates of various geometries. The above numerical methods, such as finite difference and finite element methods require fine mesh size to obtain accurate results but are computationally expensive. The method of quintic splines, characteristic orthogonal polynomials and Frobenius method require an appreciable number of terms for plates of variable thickness. Differential quadrature method (DQM), introduced by Bellman et al.[1972] which was generalized and simplified subsequently by Quan and Chang[1989a, 1989b] and Shu and Richards[1992], has emerged as a distinct numerical technique during last two decades. This method has the capability of producing highly accurate results with minimum computational efforts for initial and boundary value problems. This has led to the study of vibrational behaviour of plates of various geometries using DQ method by a number of researchers (Bert et al.[1988], Wang et al.[1993], Wang et al.[1995], Bert and Malik[1996a, 1996b], Liew et al.[1997], Malekzadeh and Shahpari[2005]), to mention a few.

This chapter deals with the axisymmetric vibrations of non-homogeneous annular plates of quadratically varying thickness using differential quadrature method. This type of thickness variation was considered earlier by Singh and Saxena[1996a] and has the advantage of dealing with linear and parabolic thickness variation, which are of practical importance. The non-homogeneity occurs due to variation in Young's modulus as well as density, which have been assumed to vary exponentially in radial direction.

2. BASIC PLATE EQUATION

Consider an isotropic annular plate of thickness $h = h(r)$ with inner and outer radii b and a, respectively, referred to a system of cylindrical coordinates (r, θ, z), where the axis of the plate is taken as the line $r = 0$ and its middle surface as the plane $z = 0$, shown in Figure 2.

Strain Displacement Relations

Let (u, v, w) be the displacement components at a point (r, θ, z) in r, θ and z directions, respectively. We assume that u and v are proportional to z, and w is independent of z. For axisymmetric vibrations, the displacement will also be axisymmetric and hence $\frac{\partial}{\partial\theta}(\ ) = 0$.

Therefore, displacement components are given by

$$\begin{aligned} u &= -z\frac{\partial w}{\partial r}, \\ v &= 0, \\ w &= w(r,t), \end{aligned} \tag{2.2.1}$$

and strain components in terms of displacements, become

$$\varepsilon_r = -z\frac{\partial^2 w}{\partial r^2},$$
$$\varepsilon_\theta = -\frac{z}{r}\frac{\partial w}{\partial r}, \tag{2.2.2}$$
$$\varepsilon_{r\theta} = \varepsilon_{\theta z} = 0 .$$

Stress-Strain Relations

The stress-strain relations for the isotropic material are given by

$$\sigma_r = \frac{E}{(1-\upsilon^2)}[\varepsilon_r + \upsilon\varepsilon_\theta],$$
$$\sigma_\theta = \frac{E}{(1-\upsilon^2)}[\varepsilon_\theta + \upsilon\varepsilon_r], \tag{2.2.3}$$
$$\sigma_{rz} = 0,\ \sigma_{r\theta} = 0 ,$$

where E is the Young's modulus of elasticity and υ the Poisson's ratio.

Using relations (2.2.2) in (2.2.3), we get,

$$\sigma_r = -\frac{E\,z}{(1-\upsilon^2)}\left\{\frac{\partial^2 w}{\partial r^2} + \upsilon\frac{1}{r}\frac{\partial w}{\partial r}\right\},$$
$$\sigma_\theta = -\frac{E\upsilon z}{1-\upsilon^2}\left\{\frac{\partial^2 w}{\partial r^2} + \frac{1}{\upsilon}\left(\frac{1}{r}\frac{\partial w}{\partial r}\right)\right\}, \tag{2.2.4}$$

$$\sigma_{r\theta} = 0.$$

If $M_r, M_{r\theta}, M_\theta$ denote the moment resultants all per unit length, then

$$(M_r, M_{r\theta}, M_\theta) = \int_{-h/2}^{h/2}(\sigma_r, \sigma_{r\theta}, \sigma_\theta)\, z\, dz . \tag{2.2.5}$$

Integration after substituting σ_r, σ_θ and $\sigma_{r\theta}$ from equation (2.2.4) into equation (2.2.5) leads to.

$$M_r = -D\left(\frac{\partial^2 w}{\partial r^2} + \frac{\upsilon}{r}\frac{\partial w}{\partial r}\right),$$
$$M_\theta = -D\left(\frac{1}{r}\frac{\partial w}{\partial r} + \upsilon\frac{\partial^2 w}{\partial r^2}\right), \tag{2.2.6}$$
$$M_{r\theta} = 0\ ,$$

where D is the flexural rigidity defined by

$$D = \frac{E h^3}{12\left(1-\upsilon^2\right)}.$$

Energy Variations

The strain energy density is given by

$$dW = \frac{1}{2}\left[\sigma_r\varepsilon_r + \sigma_\theta\varepsilon_\theta + \sigma_{r\theta}\varepsilon_{r\theta} + \sigma_{rz}\varepsilon_{rz}\right] dV\ , \tag{2.2.7}$$

where dV denotes elementary volume.

The total strain energy of the plate is obtained by integrating relation (2.2.7) over the total volume of the plate. This gives

$$W = \frac{1}{2}\int_b^a\int_0^{2\pi}\int_{-h/2}^{h/2}\left(\sigma_r\varepsilon_r + \sigma_\theta\varepsilon_\theta\right) r\, dz\, d\theta\, dr\ . \tag{2.2.8}$$

Substituting the values of ε_r, $\varepsilon_\theta, \sigma_r$, σ_θ from equations (2.2.2) and (2.2.4) in equation (2.2.8), we get

$$W = \frac{1}{2}\int_b^a\int_0^{2\pi}\int_{-h/2}^{h/2}\frac{E}{\left(1-\upsilon^2\right)}\left[\left(\frac{\partial^2 w}{\partial r^2}\right)^2 + \frac{2\upsilon}{r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\left(\frac{\partial w}{\partial r}\right)^2\right] z^2 r\, dz\, d\theta\, dr\ . \tag{2.2.9}$$

Now integrating with respect to z, it leads to

$$W = \frac{1}{2}\int_b^a \int_0^{2\pi} D\left[\left(\frac{\partial^2 w}{\partial r^2}\right)^2 + \frac{2\upsilon}{r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\left(\frac{\partial w}{\partial r}\right)^2\right] r\, d\theta\, dr \; . \tag{2.2.10}$$

The expression for kinetic energy is given by

$$dT = \frac{\rho}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] dV \; . \tag{2.2.11}$$

The total kinetic energy, resulting from the vertical movement of the elements of the plate, is given by

$$T = \frac{1}{2}\int_b^a \int_0^{2\pi} \int_{-h/2}^{h/2} \rho \left(\frac{\partial w}{\partial t}\right)^2 r\, dz\, d\theta\, dr \; . \tag{2.2.12}$$

Now integrating with respect to z, we get

$$T = \frac{1}{2}\int_b^a \int_0^{2\pi} \rho h \left(\frac{\partial w}{\partial t}\right)^2 r\, d\theta\, dr \; . \tag{2.2.13}$$

Taking variations of W and T

$$\delta W = \int_b^a \int_0^{2\pi} D\left[\frac{\partial^2 w}{\partial r^2}\frac{\partial^2 (\delta w)}{\partial r^2} + \frac{\upsilon}{r}\left(\frac{\partial w}{\partial r}\frac{\partial^2 (\delta w)}{\partial r^2} + \frac{\partial (\delta w)}{\partial r}\frac{\partial^2 w}{\partial r^2}\right) + \frac{1}{r^2}\frac{\partial w}{\partial r}\frac{\partial (\delta w)}{\partial r}\right] r\, d\theta\, dr \; , \tag{2.2.14}$$

$$\delta T = \int_b^a \int_0^{2\pi} \rho h \left(\frac{\partial w}{\partial t}\frac{\partial (\delta w)}{\partial t}\right) r\, d\theta\, dr \; . \tag{2.2.15}$$

Equation of Motion

To obtain equations of motion, Hamilton's energy principle is used which can be written as,

$$\delta \int_{t_1}^{t_2} L\, dt = 0 \ , \tag{2.2.16}$$

where t_1 and t_2 are the initial and final values of time t and the kinetic potential L is given by

$L = T - W$.

Taking the variational operator δ inside the integral and substituting from equations (2.2.14) and (2.2.15) and considering δW-δT, we get

$$\int_b^a \int_0^{2\pi} \int_{t_1}^{t_2} \left[D \left\{ \begin{array}{l} \dfrac{\partial^2 w}{\partial r^2} \dfrac{\partial^2 (\delta w)}{\partial r^2} + \dfrac{\upsilon}{r} \left(\dfrac{\partial (\delta w)}{\partial r} \dfrac{\partial^2 w}{\partial r^2} + \right. \\ \left. \dfrac{\partial w}{\partial r} \dfrac{\partial^2 (\delta w)}{\partial r^2} \right) + \dfrac{1}{r^2} \dfrac{\partial w}{\partial r} \dfrac{\partial (\delta w)}{\partial r} \end{array} \right\} - \rho h \dfrac{\partial w}{\partial t} \dfrac{\partial (\delta w)}{\partial t} \right] r\, dt\, d\theta\, dr = 0 \ . \tag{2.2.17}$$

Integrating equation (2.2.17) by parts, the integrated parts give boundary conditions while the remaining triple integrals are

$$\int_b^a \int_0^{2\pi} \int_{t_1}^{t_2} \left[\left\{ \begin{array}{l} \dfrac{\partial^2}{\partial r^2} \left(D r \dfrac{\partial^2 w}{\partial r^2} \right) - \upsilon \dfrac{\partial}{\partial r} \left(D \dfrac{\partial^2 w}{\partial r^2} \right) \\ + \upsilon \dfrac{\partial^2}{\partial r^2} \left(D \dfrac{\partial w}{\partial r} \right) - \dfrac{\partial}{\partial r} \left(\dfrac{D}{r} \dfrac{\partial w}{\partial r} \right) \end{array} \right\} + \rho h r \dfrac{\partial^2 w}{\partial t^2} \right] \delta w\, dt\, d\theta\, dr = 0 \ . \tag{2.2.18}$$

Expression (2.2.18) will be satisfied only when the coefficient of δw is zero and hence, we get

$$\left\{ \frac{\partial^2}{\partial r^2} \left(D r \frac{\partial^2 w}{\partial r^2} \right) - \upsilon \frac{\partial}{\partial r} \left(D \frac{\partial^2 w}{\partial r^2} \right) + \upsilon \frac{\partial^2}{\partial r^2} \left(D \frac{\partial w}{\partial r} \right) - \frac{\partial}{\partial r} \left(\frac{D}{r} \frac{\partial w}{\partial r} \right) \right\} + \rho h r \frac{\partial^2 w}{\partial t^2} = 0 . \tag{2.2.19}$$

Equation (2.2.19) after simplification leads to

$$D\frac{\partial^4 w}{\partial r^4}+\frac{2}{r}\left[D+r\frac{\partial D}{\partial r}\right]\frac{\partial^3 w}{\partial r^3}+\frac{1}{r^2}\left[-D+r(2+\upsilon)\frac{\partial D}{\partial r}+r^2\frac{\partial^2 D}{\partial r^2}\right]\frac{\partial^2 w}{\partial r^2}$$
$$+\frac{1}{r^3}\left[D-r\frac{\partial D}{\partial r}+r^2\upsilon\frac{\partial^2 D}{\partial r^2}\right]\frac{\partial w}{\partial r}+\rho h\frac{\partial^2 w}{\partial t^2}=0. \qquad (2.2.20)$$

For non-homogeneity of the plate material, let us assume that E and ρ are functions of space variable r. Now equation (2.2.20) becomes

$$Eh^3\frac{\partial^4 w}{\partial r^4}+\left[\frac{2}{r}\left\{Eh^3+r\left(h^3\frac{dE}{dr}+3Eh^2\frac{dh}{dr}\right)\right\}\right]\frac{\partial^3 w}{\partial r^3}$$
$$+\left[\frac{1}{r^2}\left\{\begin{aligned}&-Eh^3+r(2+\upsilon)\left(h^3\frac{dE}{dr}+3Eh^2\frac{dh}{dr}\right)\\&+r^2\left(h^3\frac{d^2E}{dr^2}+6h^2\frac{dE}{dr}\frac{dh}{dr}+3E\left(2h\left(\frac{dh}{dr}\right)^2+h^2\frac{d^2h}{dr^2}\right)\right)\end{aligned}\right\}\right]\frac{\partial^2 w}{\partial r^2}$$
$$+\left[\frac{1}{r^3}\left\{Eh^3-r\left(h^3\frac{dE}{dr}+3Eh^2\frac{dh}{dr}\right)+r^2\upsilon\left(\begin{aligned}&h^3\frac{d^2E}{dr^2}+6h^2\frac{dE}{dr}\frac{dh}{dr}\\&+3E\left(2h\left(\frac{dh}{dr}\right)^2+h^2\frac{d^2h}{dr^2}\right)\end{aligned}\right)\right\}\right]\frac{\partial w}{\partial r}$$
$$+12\rho h(1-\upsilon^2)\frac{\partial^2 w}{\partial t^2}=0. \qquad (2.2.21)$$

Introducing the non-dimensional variables $x=\frac{r}{a}$, $\overline{w}=\frac{w}{a}$, $\overline{h}=\frac{h}{a}$ together with the quadratic variation in thickness i.e.

$$\overline{h}=h_0(1+\alpha x+\beta x^2),\ \text{such that } |\alpha|\le 1,\ |\beta|\le 1 \text{ and } \alpha+\beta>-1, \qquad (2.2.22)$$

and assuming the exponential variation (following Tomar et al.[1982a, 1982b, 1983, 1984]) for the non-homogeneity of material in radial direction as follows :

$$E=E_0e^{\mu x}, \qquad \rho=\rho_0 e^{\eta x}, \qquad (2.2.23)$$

equation (2.2.21) reduces to

$$P_0 \frac{d^4W}{dx^4} + P_1 \frac{d^3W}{dx^3} + P_2 \frac{d^2W}{dx^2} + P_3 \frac{dW}{dx} + P_4 W = 0, \tag{2.2.24}$$

where, $\overline{w}(x,t) = W(x)e^{i\omega t}$ (for harmonic vibrations), ω is the radian frequency, h_0, ρ_0 are the thickness and density at the centre of the plate, μ and η are non-homogeneity parameters, α and β are the taper parameters and

$$P_0 = 1, \qquad P_1 = \frac{2}{x}(1 + \beta x),$$

$$P_2 = B^2 + C + (2+\nu)\frac{B}{x} - \frac{1}{x^2}, \qquad P_3 = \frac{1}{x^3}(1 - Bx) + \frac{\nu}{x}(B^2 + C), \tag{2.2.25}$$

$$P_4 = -\frac{\Omega^2 e^{(\eta-\mu)x}}{A^2}, \qquad \Omega^2 = \frac{12\rho_0 a^2 \omega^2 (1-\nu^2)}{E_0 h_0^2},$$

$$A = 1 + \alpha x + \beta x^2, \qquad B = \mu + \frac{3(\alpha + 2\beta x)}{A}, \qquad C = \frac{3(2\beta - \alpha^2 - 2\beta^2 x^2 - 2\alpha\beta x)}{A^2}$$

and E_0 is Young's modulus of plate material at $x = 0$.

Equation (2.2.24), which is a fourth order linear differential equation with variable coefficients involving several plate parameters, becomes quite complex and so its exact solution is not possible. Equation (2.2.24) together with the boundary conditions at the edges $x = \varepsilon$ and $x = 1$, where $\varepsilon = b/a$, constitutes a two point boundary value problem in the range $(\varepsilon, 1)$, and has been solved by differential quadrature method (DQM).

3. METHOD OF SOLUTION : DIFFERENTIAL QUADRATURE METHOD

Let $x_1, x_2, \ldots, x_m$ be the m grid points in the applicability range $[\varepsilon, 1]$. According to differential quadrature method (Bert et al.[1988]), the n^{th} order derivative of $W(x)$ with respect to x can be expressed discretely at the point x_i as

$$\frac{d^n W(x_i)}{dx^n} = \sum_{j=1}^{m} c_{ij}^{(n)} W(x_j), \qquad n = 1,2,3,4 \qquad i = 1,2,\ldots,m \tag{2.3.1}$$

where $c_{ij}^{(n)}$ are weighting coefficients associated with n^{th} order derivative of $W(x)$ with respect to x at discrete point x_i.

Following Shu [2000; pages 31, 35], the weighting coefficients in equation (2.3.1) are given by

$$c_{ij}^{(1)} = \frac{M^{(1)}(x_i)}{(x_i - x_j) M^{(1)}(x_j)}, \qquad i, j = 1,2,\ldots,m; \qquad i \neq j \tag{2.3.2}$$

where

$$M^{(1)}(x_i) = \prod_{\substack{j=1 \\ j \neq i}}^{m} (x_i - x_j) \tag{2.3.3}$$

and

$$c_{ij}^{(n)} = n\left(c_{ii}^{(n-1)} c_{ij}^{(1)} - \frac{c_{ij}^{(n-1)}}{x_i - x_j} \right), \qquad i, j = 1,2,\ldots,m; \quad j \neq i; \qquad n = 2,3,4 \tag{2.3.4}$$

$$c_{ii}^{(n)} = -\sum_{\substack{j=1 \\ j \neq i}}^{m} c_{ij}^{(n)}, \qquad i = 1,2,\ldots,m; \qquad n = 1,2,3,4 \tag{2.3.5}$$

Discretizing the equation (2.2.24) at nodes x_i, $i = 3, 4, \ldots, m-2$, equation (2.2.24) reduces to,

$$P_0 \frac{d^4 W(x_i)}{dx^4} + P_{1,i} \frac{d^3 W(x_i)}{dx^3} + P_{2,i} \frac{d^2 W(x_i)}{dx^2} + P_{3,i} \frac{dW(x_i)}{dx} + P_{4,i} W(x_i) = 0, \qquad i = 3,4,\ldots,(m-2) \tag{2.3.6}$$

Substituting the expressions for first four derivatives at node x_i in equation (2.3.6) using relations (2.3.1) to (2.3.5), the equation (2.3.6) becomes

$$\sum_{j=1}^{m}(P_0 c_{ij}^{(4)} + P_{1,i} c_{ij}^{(3)} + P_{2,i} c_{ij}^{(2)} + P_{3,i} c_{ij}^{(1)})W(x_j) + P_{4,i} W(x_i) = 0, \qquad i = 3,4,\ldots,(m-2). \qquad (2.3.7)$$

The satisfaction of equation (2.3.7) at (m-4) nodal points x_i, i = 3, 4,..., (m-2) provides a set of (m-4) equations in terms of unknowns W_j, j = 1, 2,..., m (where W_j stands for $W(x_j)$), which can be written in matrix form as

$$[B][W^*] = [0] \ , \qquad (2.3.8)$$

where B and W^* are matrices of order (m-4) x m and m x 1, respectively.

Here, the (m-2) internal grid points, chosen for collocation, are the zeros of shifted Chebyshev polynomial of order (m-2) with orthogonality range (ε, 1) given by

$$x_{k+1} = \frac{1}{2}\left[(1+\varepsilon)+(1-\varepsilon)\cos\left(\frac{2k-1}{m-2}\frac{\pi}{2}\right)\right], \qquad k = 1,2,\ldots,(m-2). \qquad (2.3.9)$$

However, for a specified plate, the following three different sets of grid points have also been considered for a comparative study :

(i) zeros of shifted Legendre polynomial $P_n^*(x)$ (Bellman et al.[1972]), satisfying the differential equation

$$x(1-x)P_n^{*''}(x)+(1-2x)P_n^{*'}+n(n+1)P_n^*(x)=0,$$

(ii) grid points taken by Liew et al.[1997]

$$x_k = \frac{1}{2}\left[1-\cos\frac{(k-1)\pi}{m-1}\right], \qquad k=1, 2, \ldots, m \ , \qquad (2.3.10)$$

(iii) equally spaced grid points (Bert et al.[1988]).

4. BOUNDARY CONDITIONS AND FREQUENCY EQUATIONS

The three different combinations of boundary conditions have been considered which are

(a) C-C, (b) C-S and (c) C-F ,

where first suffix denotes the boundary condition at inner edge and second at the outer edge. The symbols C, S and F are used for clamped, simply-supported and free.

By satisfying the relations :

(i) $W = \dfrac{dW}{dx} = 0$

(ii) $W = \dfrac{d^2W}{dx^2} + \dfrac{\upsilon}{x}\dfrac{dW}{dx} = 0$

(iii) $\dfrac{d^2W}{dx^2} + \dfrac{\upsilon}{x}\dfrac{dW}{dx} = \dfrac{d^3W}{dx^3} + \dfrac{1}{x}\dfrac{d^2W}{dx^2} - \dfrac{1}{x^2}\dfrac{dW}{dx} = 0$

for clamped, simply-supported and free edge conditions respectively, a set of four homogeneous equations in terms of W_j are obtained. These equations together with field equations (2.3.8) give a complete set of m equations in m unknowns. For a C-C plate, the above set of homogeneous equations can be written as

$$\begin{bmatrix} B \\ B^{CC} \end{bmatrix} \left[W^{*} \right] = [0] , \tag{2.4.1}$$

where B^{CC} is a matrix of order 4 x m.

For a non-trivial solution of equation (2.4.1), the frequency determinant must vanish and hence

$$\begin{vmatrix} B \\ B^{CC} \end{vmatrix} = 0 \; . \tag{2.4.2}$$

Similarly for C-S and C-F plates, the frequency determinants can respectively be written as

$$\begin{vmatrix} B \\ B^{CS} \end{vmatrix} = 0, \tag{2.4.3}$$

$$\begin{vmatrix} B \\ B^{CF} \end{vmatrix} = 0 . \tag{2.4.4}$$

5. NUMERICAL RESULTS AND DISCUSSION

The frequency equations (2.4.2-2.4.4) provide the values of the frequency parameter Ω for various values of plate parameters. The first three natural frequencies of vibration have been computed for three different combinations of boundary conditions i.e. C-C, C-S and C-F for non-homogeneity parameter μ = -0.5(0.1)1.0; density parameter η = -0.5(0.1)1.0; radii ratio ε = 0.1(0.05)0.7 and taper parameters α = -0.5(0.1)0.5; β = -0.5(0.1)0.5 such that $\alpha+\beta>-1$ for $\upsilon = 0.3$.

To choose appropriate number of grid points m, convergence studies have been carried out for different sets of plate parameters. The normalized frequency parameter Ω/Ω^* for first three modes of vibration for specified plate i.e. μ = -0.5, η = 1.0, α = -0.2, β = -0.3, ε = 0.3 are presented in Figures 2.1(a,b,c) for C-C, C-S and C-F plates, respectively. It is observed that frequency parameter converges with increasing number of grid points. For convergence of frequency parameter in higher modes more grid points are needed than for the lower ones. The convergence of frequency with increasing number of grid points is oscillatory for C-C and C-S plate while for C-F plate it is monotonic. The value of m has been fixed as 20, since there was no further improvement even in the fourth place of decimal for all the three plates. Calculations were carried out with double precision arithmetic on Pentium-IV.

The numerical results for specified plate parameters are given in Tables (2.1-2.12) and Figures (2.2-2.7). Tables (2.1-2.12) present the values of frequency parameter Ω for different values of plate parameters i.e. density parameter η (= -0.5, 0.0, 1.0), non-homogeneity parameter μ (= -0.5, 0.0, 1.0), taper parameters α (= -0.5, 0.0, 0.5); β (= -0.5, 0.0, 0.5) such that $\alpha + \beta > -1$ for radii ratio ε (= 0.1, 0.3, 0.5, 0.7) for C-C, C-S and C-F plates, respectively. From the results, it is found that the frequency parameter for C-S plate is smaller than that for C-C plate and greater than that for C-F plate irrespective of the values of other plate parameters. The value of frequency parameter Ω increases with increasing values of non-homogeneity parameter μ, taper parameters α and β and radii ratio ε, while it decreases with increasing values of density parameter η.

Figures 2.2(a,b,c) show the effect of non-homogeneity parameter μ on the frequency parameter Ω for $\eta = 0.5$, $\varepsilon = 0.3$, $\alpha = 0.0, 0.3$ and $\beta = 0.0, \pm 0.3$ for all the three plates vibrating in fundamental, second and third mode, respectively. It is observed that frequency parameter increases with increasing values of non-homogeneity parameter μ for all the three plates. The rate of increase of Ω with μ is more pronounced in the case of C-C plate as compared to C-S and C-F plates. The frequency parameter increases with increasing values of α or β or both for all the three plates. Also, the rate of increase of Ω with increasing values of μ in all the three plates becomes higher and higher with increase in the number of modes.

Figures 2.3(a,b,c) depict the variation of frequency parameter Ω with density parameter η for $\mu = 0.5$, $\varepsilon = 0.3$, $\alpha = 0.0, 0.3$ and $\beta = 0.0, \pm 0.3$ for all the three plates vibrating in fundamental, second and third mode, respectively. It is seen that frequency parameter Ω decreases with

increasing value of density parameter η. The rate of decrease of Ω with increasing value of η is more pronounced in the case of C-C plate as compared to C-S and C-F plate, whatever are the values of other plate parameters. The rate of decrease in second mode is higher as compared to that in fundamental mode and the rate of decrease in third mode is higher than that in second mode.

Figures 2.4(a,b,c) show the effect of taper parameter α on frequency parameter Ω for $\mu = 0.5$, $\eta = 0.5$, $\varepsilon = 0.3, 0.5$ and $\beta = -0.3, 0.3$ for plates vibrating in fundamental, second and third modes, respectively. It is observed that frequency parameter increases with increasing value of taper parameter α for all the three plates except for C-F plate for $\varepsilon = 0.3$, $\beta = -0.3$. In this case, frequency first decreases and then increases with local minima in the vicinity of $\alpha = -0.4$. The rate of increase of Ω is higher for C-C plate as compared to those of C-S and C-F plates. Further, the frequency parameter can be increased / decreased by increasing / decreasing the value of ε as well as β except for C-F plate for $\varepsilon = 0.3$ for $\alpha < -0.1$. In that case, frequency decreases by increasing β. Also, the rate of increase of Ω with α increases with the increase in number of modes.

Figures 2.5(a,b,c) show the plots of frequency parameter Ω versus taper parameter β for $\mu = 0.5$, $\eta = 0.5$, $\varepsilon = 0.3, 0.5$ and taper parameter $\alpha = -0.3, 0.3$ for plates vibrating for first three modes of vibration, respectively. It is found that frequency parameter increases with increasing value of taper parameter β for C-C and C-S plates, irrespective of the value of other plate parameters. However, in the case of C-F plate, the frequency parameter first decreases and then increases i.e. it has a local minima in the vicinity of $\beta = 0.0$, which shifts towards lower value

of β with the increase in hole size as well as taper parameter α. In particular, for $\alpha = -0.3$, there is a local minima in the vicinity of $\beta = 0.2$ for $\varepsilon = 0.3$ and at $\beta = -0.2$ for $\varepsilon = 0.5$. The rate of increase of Ω with increasing value of β is more pronounced in the case of C-C plate as compared to those in the case of C-S and C-F plates. Also, the rate of increase of Ω with β increases with the increase in number of modes.

Figure 2.6 depicts the effect of radii ratio ε on frequency parameter Ω for $\mu = 0.5$, $\eta = 0.5$. $\alpha = -0.3$, 0.3 and $\beta = -0.3$, 0.3 for all the three plates vibrating in fundamental mode. It is observed that frequency parameter increases with increasing value of ε. The rate of increase of Ω for $\varepsilon > 0.5$ is much higher as compared to that for $\varepsilon < 0.5$ for all the three boundary conditions. This rate of increase reduces in the order of boundary conditions C-C, C-S, C-F for the same set of other plate parameters.

Figures 2.7(a,b,c) show the plots for normalized transverse displacements for $\mu = -0.5$, 1.0. $\eta = 0.5$, $\varepsilon = 0.3$, $\alpha = 0.0$, $\beta = 0.0$; $\alpha = 0.5$, $\beta = 0.0$ and $\alpha = 0.5$, $\beta =$ 0.5 for the first three modes of vibration for C-C, C-S and C-F plates, respectively. It is seen that the radii of nodal circles increase with the increasing value of non-homogeneity parameter μ as well as the inner edge becomes thicker and thicker for all the three boundary conditions.

A comparison of results with those available in literature has been presented in Tables 2.13-2.15. Table 2.13 shows a comparison of results for homogeneous ($\mu = 0.0$, $\eta = 0.0$) annular plate of uniform thickness ($\alpha = 0.0$, $\beta = 0.0$) with exact solutions given by Leissa[1969] and approximate solutions obtained by Sharma[2006] using Chebyshev collocation method.

Selmane and Lakis[1999] using finite element method and Verma[1987] using quintic splines method. Table 2.14 compares the results for homogeneous ($\mu = 0.0$, $\eta = 0.0$) annular plate of linearly varying thickness (α = -0.5(0.2)0.5, $\beta = 0.0$) with the results obtained by Lal[1979] using Chebyshev collocation method, Chen[1997] using finite element method and Verma[1987] using quintic splines method for C-C and C-S plate. A comparison of the results for homogeneous ($\mu = 0.0$, $\eta = 0.0$) annular plate of parabolically varying thickness ($\alpha = 0.0$, β = -0.5(0.2)0.5) with those obtained by Lal[1979] employing Chebyshev collocation method for C-C and C-S plates is given in Table 2.15. An excellent agreement of the results shows the versatility of present technique.

A comparative study for evaluation of frequency parameter Ω for a specified plate for the first three modes of vibration has been presented in Table 2.16 by taking equally spaced and three unequally spaced grid points i.e. zeros of shifted Chebyshev polynomials obtained from equations (2.3.9) and (2.3.10) and that of shifted Legendre polynomials. During numerical computation, it is found that for uniform grid spacing, the number of grid points is considerably greater as compared to that for non-uniform grid spacing. It is worth noting that in the case of uniform grid points, the results converge with increasing value of m up to a certain extent and after that results become unstable. This may be attributed to round-off errors. It is observed that the number of grid points taken as zeros of shifted Chebyshev polynomials (used in the present investigation) do not exceed the number of grid points as taken by Liew et al.[1997] and Bellman et al.[1972]. Thus the present choice of grid points, not only provides a comparatively faster rate of convergence, but also leads to reliable results.

Table 2.1

Values of frequency parameter Ω for C-C plate for $\varepsilon = 0.1$

		η								
		-0.5			0			1		
		μ			μ			μ		
α	β	-0.5	0	1	-0.5	0	1	-0.5	0	1
		I								
-0.5	0	19.0782	21.8774	28.7371	16.4756	18.9464	25.0280	12.1832	14.0886	18.8205
	0.5	25.6764	29.5466	39.1089	22.3682	25.8141	34.3671	16.8315	19.5364	26.3129
	-0.5	20.5152	23.5209	30.8700	17.7096	20.3608	26.8715	13.0836	15.1252	20.1831
0	0	27.1730	31.2494	41.3143	23.6546	27.2806	36.2730	17.7720	20.6123	27.7202
	0.5	33.0828	38.1462	50.7140	28.9495	33.4770	44.7663	21.9775	25.5617	34.5835
	-0.5	28.6639	32.9439	43.4964	24.9364	28.7402	38.1597	18.7094	21.6838	29.1149
0.5	0	34.6148	39.8862	52.9688	30.2668	34.9756	46.7144	22.9408	26.6610	36.0206
	0.5	40.1922	46.4114	61.8971	35.2738	40.8500	54.7991	26.9338	31.3728	42.5832
		II								
-0.5	0	52.8251	60.7775	80.1698	45.6748	52.6801	69.8322	33.8944	39.2840	52.5882
	0.5	69.6207	79.9665	105.1286	60.6503	69.8328	92.2532	45.6914	52.8652	70.5207
	-0.5	58.0039	66.7389	88.0460	50.1377	57.8312	76.6753	37.1793	43.0959	57.7074
0	0	75.1661	86.3503	113.5555	65.4435	75.3663	99.6000	49.2401	56.9852	76.0544
	0.5	89.8601	103.1224	135.3243	78.5887	90.4082	119.2189	59.6669	68.9777	91.8534
	-0.5	80.5974	92.6045	121.8190	70.1353	80.7840	106.7992	52.7099	61.0139	81.4691
0.5	0	95.5669	109.6934	144.0004	83.5276	96.1115	126.7936	63.3322	73.2345	97.5742
	0.5	109.2116	125.2574	164.1721	95.7592	110.0986	145.0095	73.0755	84.4336	112.3049
		III								
-0.5	0	104.1922	120.0242	158.5007	90.1199	104.0538	138.0505	66.9557	77.6636	103.9955
	0.5	135.8939	155.9447	204.3934	118.3704	136.1443	179.2520	89.1990	103.0588	136.9276
	-0.5	115.5359	133.1071	175.7974	99.9058	115.3715	153.0979	74.1787	86.0637	115.2899
0	0	147.9703	169.8501	222.7151	128.8230	148.2137	195.2444	96.9651	112.0765	149.0119
	0.5	175.3996	200.8708	262.1995	153.3474	176.0177	230.8044	116.4075	134.2294	177.6226
	-0.5	159.8180	183.4965	240.7079	139.0723	160.0523	210.9418	104.5721	120.9124	160.8612
0.5	0	187.7977	215.1368	280.9694	164.0944	188.4187	247.2129	124.4153	143.5220	190.0588
	0.5	213.0875	243.7039	317.2455	186.7518	214.0747	279.9465	142.4529	164.0485	216.5015

Table 2.2
Values of frequency parameter Ω for C-C plate for $\varepsilon = 0.3$

		η								
		-0.5			0			1		
		μ			μ			μ		
α	β	-0.5	0	1	-0.5	0	1	-0.5	0	1
I										
-0.5	0	29.8639	35.1222	48.5495	25.2335	29.7276	41.2340	17.9224	21.1864	29.5887
	0.5	41.9983	49.5019	68.7403	35.7121	42.1662	58.7592	25.6880	30.4364	42.7120
0	-0.5	32.8105	38.5799	53.2999	27.6949	32.6202	45.2198	19.6296	23.1986	32.3772
	0	45.2406	53.2976	73.9349	38.4248	45.3462	63.1230	27.5750	32.6543	45.7712
	0.5	56.5277	66.6883	92.7847	48.1855	56.9460	79.5089	34.8303	41.3069	58.0809
0.5	-0.5	48.4295	57.0322	79.0459	41.0922	48.4746	67.4160	29.4297	34.8354	48.7804
	0	59.9148	70.6509	98.2027	51.0210	60.2677	84.0623	36.8046	43.6256	61.2751
	0.5	70.8150	83.5909	116.4417	60.4540	71.4856	99.9300	43.8273	52.0067	73.2157
II										
-0.5	0	82.8038	97.5741	135.1833	70.0158	82.6289	114.8228	49.8348	58.9884	82.4649
	0.5	114.7703	135.0557	186.5973	97.5812	114.9999	159.3617	70.2255	83.0083	115.7177
0	-0.5	92.2704	108.7509	150.7274	77.9522	92.0144	127.9184	55.3841	65.5721	91.7111
	0	125.2016	147.3683	203.7097	106.3456	125.3621	173.8117	76.3799	90.3086	125.9659
	0.5	154.6659	181.9028	251.0422	131.7859	155.2258	214.8658	95.2498	112.5280	156.6985
0.5	-0.5	135.4076	159.4160	220.4582	114.9162	135.4958	187.9462	82.3916	97.4400	135.9786
	0	165.5379	194.7346	268.8736	140.9273	166.0329	229.9343	101.6775	120.1522	167.4004
	0.5	193.8493	227.9109	314.3244	165.3884	194.7409	269.3815	119.8468	141.5418	196.9706
III										
-0.5	0	162.8812	192.0798	266.3226	137.7694	162.6951	226.2176	98.1500	116.2328	162.5270
	0.5	224.2744	263.7248	363.6012	190.6768	224.5293	310.4221	137.2525	162.0701	225.3184
0	-0.5	182.6486	215.4837	299.0142	154.3641	182.3743	253.7956	109.7892	130.0796	182.0572
	0	245.9753	289.3880	399.3650	208.9360	246.1573	340.6606	150.1161	177.3563	246.8294
	0.5	302.3385	355.1183	488.4935	257.5676	302.9540	417.8973	186.1510	219.5657	304.5642
0.5	-0.5	267.2234	314.5232	434.4141	226.8042	267.3284	370.2780	162.6901	192.3026	267.8748
	0	324.9222	381.8152	525.6695	276.5837	325.4702	449.3538	199.5669	235.5025	326.9747
	0.5	378.9667	444.8192	611.0422	323.2438	379.9443	523.3785	234.1865	276.0382	382.3769

Table 2.3
Values of frequency parameter Ω for C-C plate for ε = 0.5

		η								
		-0.5			0			1		
		μ			μ			μ		
α	β	-0.5	0	1	-0.5	0	1	-0.5	0	1
I										
-0.5	0	55.0406	66.3711	96.4768	45.4803	54.8912	79.9310	30.9707	37.4451	54.7192
	0.5	82.6856	99.8274	145.4660	68.5852	82.8778	120.9834	47.0626	56.9715	83.4640
0	-0.5	60.8822	73.3962	106.6310	50.2552	60.6381	88.2508	34.1511	41.2789	60.2866
	0	89.1591	107.6079	156.6965	73.8837	89.2508	130.1962	50.6006	61.2334	89.6431
	0.5	115.8277	139.8897	203.9925	96.1804	116.2646	169.8450	66.1417	80.0961	117.4281
0.5	-0.5	95.4742	115.1996	167.6585	79.0509	95.4673	139.1864	54.0490	65.3883	95.6699
	0	122.5308	147.9452	215.6169	101.6689	122.8654	179.3843	69.8092	84.5134	123.8304
	0.5	148.7924	179.7381	262.2092	123.6288	149.4744	218.4497	85.1208	103.0999	151.2170
II										
-0.5	0	152.1449	183.6653	267.3373	125.7716	151.9449	221.5067	85.7497	103.7533	151.7182
	0.5	226.8497	273.6121	397.5796	188.1494	227.1080	330.5110	129.1321	156.1094	227.8836
0	-0.5	169.6359	204.8257	298.2691	140.1050	169.2995	246.9172	95.3497	115.3957	168.8209
	0	246.2130	297.0313	431.7954	204.0397	246.3428	358.6601	139.8044	169.0492	246.8832
	0.5	318.0874	383.5626	557.0690	264.0721	318.6732	463.5359	181.5828	219.4646	320.2102
0.5	-0.5	265.0816	319.8538	465.1442	219.5181	265.0800	386.0850	150.1921	181.6446	265.3795
	0	338.1189	407.7882	592.4579	280.5172	338.5780	492.6616	192.6358	232.8650	339.8836
	0.5	408.8108	492.8926	715.6553	339.5698	409.7249	595.8123	233.7441	282.4696	412.0243
III										
-0.5	0	298.6657	360.7037	525.3153	246.9403	298.4481	435.2742	168.4502	203.8782	298.2031
	0.5	443.8486	535.1213	776.6651	368.1161	444.1320	645.5241	252.6705	305.2823	444.9805
0	-0.5	334.1550	403.7500	588.5360	276.0525	333.7878	487.2601	187.9923	227.6383	333.2676
	0	483.0798	582.6704	846.3941	400.3433	483.2237	702.9437	274.3633	331.6381	483.8187
	0.5	622.6037	750.2640	1087.8383	516.8282	623.2453	904.9586	355.3718	429.1580	624.9250
0.5	-0.5	521.3217	629.0301	914.4073	431.7455	521.3225	758.9282	295.4843	357.3045	521.6564
	0	663.1699	799.4212	1159.8965	550.1646	663.6744	964.3190	377.8274	456.4354	665.1059
	0.5	800.3295	964.1640	1397.2028	664.6893	801.3302	1162.8933	457.4936	552.3319	803.8422

Table 2.4
Values of frequency parameter Ω for C-C plate for ε = 0.7

		η								
		-0.5			0			1		
		μ			μ			μ		
α	β	-0.5	0	1	-0.5	0	1	-0.5	0	1
I										
-0.5	0	142.0921	175.7034	268.6238	114.7434	141.9309	217.1297	74.7526	92.5237	141.7263
	0.5	234.3102	289.8738	443.5995	189.5140	234.5300	359.1365	123.8579	153.3769	235.1685
0	-0.5	155.0014	191.6326	292.8745	125.0895	154.7012	236.5826	81.3907	100.7219	154.2292
	0	248.3238	307.1658	469.9222	200.7529	248.4021	380.2654	131.0783	162.2940	248.7658
	0.5	339.4662	420.0068	642.8706	274.6541	339.9267	520.6329	179.6170	222.4464	341.1380
0.5	-0.5	262.0211	324.0678	495.6530	211.7360	261.9590	400.9156	138.1319	171.0055	262.0505
	0	353.7662	437.6524	669.7314	286.1245	354.0847	542.1969	186.9882	231.5500	355.0193
	0.5	444.5041	549.9938	841.9187	359.6989	445.2054	681.9489	235.3139	291.4391	446.9898
II										
-0.5	0	392.0070	484.9524	741.8678	316.6111	391.7885	599.6803	206.3622	255.5030	391.5120
	0.5	644.7361	797.3167	1218.8368	521.4502	645.0328	986.5898	340.8115	421.8164	645.8902
0	-0.5	428.9663	530.7491	812.1527	346.2741	428.5551	656.1397	225.4511	279.1771	427.9089
	0	684.8840	847.0570	1295.1522	553.6934	684.9916	1047.9339	361.5875	447.5790	685.4871
	0.5	934.5195	1155.5948	1766.2692	756.0316	935.1415	1430.1080	494.4046	611.8717	936.7691
0.5	-0.5	724.1092	895.6562	1369.7225	585.1898	724.0264	1107.8639	381.8747	472.7365	724.1570
	0	975.4914	1206.3544	1844.1433	788.9413	975.9249	1492.7139	515.6159	638.1732	977.1917
	0.5	1223.9633	1513.4530	2313.0593	990.3388	1224.9110	1873.1070	647.8192	801.7059	1227.3091
III										
-0.5	0	768.7735	951.2323	1455.5445	620.9591	768.5345	1176.5943	404.8144	501.2801	768.2323
	0.5	1262.9848	1561.6195	2386.1234	1021.4588	1263.3098	1931.3068	667.6217	826.1204	1264.2482
0	-0.5	842.3921	1042.6161	1596.2683	680.0791	841.9418	1289.7006	442.9070	548.6054	841.2345
	0	1342.9642	1660.8646	2538.8479	1085.7273	1343.0825	2054.1367	709.0830	877.6134	1343.6260
	0.5	1831.0044	2263.6204	3457.7640	1481.2377	1831.6857	2799.4100	968.6332	1198.4216	1833.4669
0.5	-0.5	1421.1027	1757.8352	2688.1017	1148.5050	1421.0125	2174.1531	749.5680	927.8987	1421.1571
	0	1912.6272	2364.8974	3613.5922	1546.8356	1913.1025	2924.7532	1010.9632	1250.9898	1914.4905
	0.5	2398.3465	2964.7831	4528.1221	1940.4694	2399.3844	3666.4763	1269.2883	1570.2813	2402.0088

Table 2.5

Values of frequency parameter Ω for C-S plate for ε = 0.1

		η								
		-0.5			0			1		
		μ			μ			μ		
α	β	-0.5	0	1	-0.5	0	1	-0.5	0	1
I										
-0.5	0	14.0945	15.9801	20.4520	12.0128	13.6528	17.5567	8.6479	9.8746	12.8161
	0.5	16.7756	18.9303	23.9944	14.3622	16.2423	20.6749	10.4302	11.8456	15.2033
0	-0.5	15.5839	17.6957	22.7068	13.2932	15.1315	19.5100	9.5844	10.9616	14.2667
	0	18.3457	20.7203	26.2930	15.7159	17.7897	22.6719	11.4264	12.9901	16.6951
	0.5	20.5153	23.1029	29.1735	17.6172	19.8802	25.2041	12.8693	14.5807	18.6290
0.5	-0.5	19.8958	22.4910	28.5747	17.0517	19.3195	24.6533	12.4082	14.1203	18.1739
	0	22.1109	24.9126	31.4689	18.9946	21.4464	27.2005	13.8854	15.7419	20.1239
	0.5	24.0363	27.0313	34.0637	20.6816	23.3046	29.4792	15.1655	17.1546	21.8606
II										
-0.5	0	43.7473	50.1616	65.6707	37.6585	43.2847	56.9447	27.7104	32.0042	42.5164
	0.5	55.4001	63.2769	82.1563	48.0659	55.0350	71.8090	35.9384	41.3533	54.4991
0	-0.5	48.6688	55.8417	73.2201	41.8818	48.1719	63.4748	30.7925	35.5894	47.3571
	0	60.5319	69.2025	90.0311	52.4781	60.1435	78.6364	39.1703	45.1151	59.5826
	0.5	70.6900	80.6188	104.3285	61.5834	70.4099	91.5785	46.4236	53.3480	70.1030
0.5	-0.5	65.5770	75.0273	97.7748	56.8153	65.1643	85.3480	42.3468	48.8114	64.5770
	0	75.8966	86.6298	112.3127	66.0648	75.5979	98.5093	49.7132	57.1765	75.2757
	0.5	85.3123	97.2027	125.5247	74.5239	85.1281	110.4981	56.4837	64.8564	85.0718
III										
-0.5	0	90.8129	104.4512	137.4665	78.3728	90.3500	119.4599	57.9876	67.1557	89.6173
	0.5	116.0688	132.8310	173.0334	100.9178	115.7577	151.4900	75.7989	87.3471	115.3754
0	-0.5	101.4957	116.8013	153.8788	87.5576	100.9972	133.6872	64.7227	75.0052	100.2211
	0	127.2150	145.6934	190.0629	110.5313	126.8815	166.2978	82.8933	95.6002	126.4821
	0.5	149.0716	170.2025	220.6465	130.1068	148.8965	193.9298	98.4662	113.2171	148.8564
0.5	-0.5	138.1943	158.3664	206.8504	119.9971	137.8373	180.8905	89.8728	103.7221	137.4192
	0	160.4220	183.2877	237.9352	139.9118	160.2310	208.9878	105.7242	121.6530	160.1877
	0.5	180.5710	205.8527	266.0190	157.9951	180.5414	234.4119	120.1717	137.9754	180.8614

Table 2.6

Values of frequency parameter Ω for C-S plate for ε = 0.3

α	β	η = -0.5, μ = -0.5	η = -0.5, μ = 0	η = -0.5, μ = 1	η = 0, μ = -0.5	η = 0, μ = 0	η = 0, μ = 1	η = 1, μ = -0.5	η = 1, μ = 0	η = 1, μ = 1
					I					
-0.5	0	21.7966	25.4001	34.4067	18.2248	21.2689	28.8937	12.6709	14.8296	20.2600
	0.5	27.5965	32.0478	43.0805	23.1568	26.9280	36.2929	16.2132	18.9029	25.6078
0	-0.5	24.6425	28.7521	39.0466	20.6020	24.0735	32.7887	14.3202	16.7817	22.9887
	0	30.6950	35.6845	48.0689	25.7509	29.9777	40.4898	18.0210	21.0352	28.5611
	0.5	35.8742	41.6072	55.7757	30.1541	35.0181	47.0608	21.1826	24.6614	33.3061
0.5	-0.5	33.7394	39.2623	52.9898	28.2983	32.9763	44.6278	19.7943	23.1294	31.4699
	0	39.0813	45.3668	60.9152	32.8413	38.1733	51.3880	23.0582	26.8709	36.3557
	0.5	43.9772	50.9613	68.1932	37.0030	42.9335	57.5915	26.0457	30.2945	40.8329
					II					
-0.5	0	68.3245	80.2919	110.5832	57.5716	67.7557	93.5959	40.7035	48.0453	66.7647
	0.5	91.6160	107.3603	146.9902	77.6408	91.1204	125.1332	55.5314	65.3699	90.3191
0	-0.5	77.0726	90.6331	125.0034	64.8848	76.4146	105.7092	45.7879	54.0843	75.2669
	0	101.0028	118.4502	162.4348	85.5020	100.4228	138.1307	61.0156	71.8801	99.4741
	0.5	122.4808	143.4003	195.9585	104.0311	121.9854	167.2062	74.7417	87.9111	121.2491
0.5	-0.5	110.2397	129.3639	177.6378	93.2348	109.5739	150.9197	66.4065	78.2798	108.4751
	0	132.1634	154.8352	211.8704	112.1456	131.5838	180.6065	80.4101	94.6373	130.7003
	0.5	152.8064	178.8102	244.0686	129.9655	152.3169	208.5503	93.6291	110.0732	151.6573
					III					
-0.5	0	141.7322	166.9279	230.8120	119.6727	141.1443	195.7110	84.9784	100.5056	140.1478
	0.5	191.9412	225.2457	309.1649	162.9444	191.4866	263.5701	116.9681	137.8453	190.8137
0	-0.5	160.0398	188.6265	261.1899	135.0093	159.3503	221.2805	95.6908	113.2620	158.1770
	0	211.6788	248.6033	341.7653	179.5167	211.1294	291.0727	128.5956	151.6705	210.2940
	0.5	257.8388	302.1835	413.6597	219.3431	257.4324	353.4025	158.1092	186.0954	256.9369
0.5	-0.5	231.0865	271.5783	373.8542	195.8031	230.4400	318.1296	140.0103	165.2473	229.4380
	0	278.2564	326.3317	447.3232	236.5002	277.7565	381.8257	170.1657	200.4227	277.1019
	0.5	322.5429	377.7199	516.2298	274.7322	322.1903	441.5969	198.5331	233.4984	321.8825

Table 2.7
Values of frequency parameter Ω for C-S plate for ε = 0.5

		η								
		-0.5			0			1		
		μ			μ			μ		
α	β	-0.5	0	1	-0.5	0	1	-0.5	0	1
		I								
-0.5	0	39.6941	47.5364	68.0839	32.5465	39.0056	55.9483	21.8180	26.1864	37.6703
	0.5	54.9213	65.6351	93.5726	45.1359	53.9781	77.0590	30.3957	36.4000	52.1048
0	-0.5	45.1292	54.0850	77.5879	36.9859	44.3594	63.7318	24.7715	29.7544	42.8756
	0	60.8462	72.7697	103.9049	49.9829	59.8199	85.5340	33.6303	40.3052	57.7899
	0.5	75.3621	90.0093	128.1524	61.9818	74.0786	105.6116	41.8027	50.0281	71.5116
0.5	-0.5	66.6429	79.7525	114.0269	54.7231	65.5351	93.8331	36.7910	44.1226	63.3525
	0	81.4569	97.3475	138.7734	66.9700	80.0898	114.3272	45.1345	54.0501	77.3633
	0.5	95.6754	114.2292	162.5071	78.7220	94.0511	133.9771	53.1374	63.5684	90.7897
		II								
-0.5	0	125.0411	150.6337	218.2967	103.1078	124.3055	180.4168	69.9567	84.4669	122.9695
	0.5	181.7712	218.6153	315.7372	150.4019	181.0269	261.8540	102.7516	123.8654	179.7277
0	-0.5	140.9040	169.8328	246.3880	116.0819	140.0209	203.4483	78.6130	94.9689	138.4099
	0	198.9461	239.3897	346.0964	164.4670	198.0535	286.7766	112.1592	135.2719	196.4776
	0.5	253.5987	304.8783	439.9515	210.0397	252.7074	365.2350	143.7771	173.2542	251.1881
0.5	-0.5	215.7791	259.7536	375.8649	178.2476	214.7381	311.2060	121.3705	146.4420	212.8851
	0	271.2516	326.2259	471.1341	224.5017	270.2104	390.8431	153.4572	184.9883	268.4112
	0.5	325.0357	390.6724	563.4914	269.3548	324.0000	468.0570	184.5831	222.3788	322.2663
		III								
-0.5	0	259.3575	312.9243	454.7848	214.1751	258.5950	376.3659	145.7544	176.2358	257.2359
	0.5	380.6252	458.2599	663.1366	315.3313	379.9217	550.5688	215.9868	260.6043	378.7528
0	-0.5	291.8464	352.3468	512.7314	240.7904	290.9158	423.9489	163.5737	197.9086	289.2433
	0	416.0283	501.1745	726.0872	344.3759	415.1567	602.3365	235.4879	284.2988	413.6757
	0.5	532.7079	640.9961	926.4917	441.7266	531.9052	769.9272	303.1119	365.5229	530.6324
0.5	-0.5	450.6690	543.1765	787.7330	372.7839	449.6286	653.0109	254.5464	307.4620	447.8340
	0	569.1777	685.1907	991.2811	471.6584	568.2058	823.2291	323.2242	389.9532	566.6192
	0.5	683.9349	822.7019	1188.3558	567.4145	683.0359	988.0491	389.7552	469.8596	681.6648

Table 2.8
Values of frequency parameter Ω for C-S plate for ε = 0.7

		η								
		-0.5			0			1		
		μ			μ			μ		
α	β	-0.5	0	1	-0.5	0	1	-0.5	0	1
		I								
-0.5	0	100.9643	124.2931	188.2697	81.1395	99.9142	151.4227	52.3489	64.4959	97.8481
	0.5	157.7340	194.0143	293.3384	126.8905	156.1162	236.1575	82.0307	100.9750	152.8979
0	-0.5	112.6722	138.7531	210.3261	90.5160	111.4985	169.1042	58.3566	71.9230	109.1990
	0	170.2634	209.4910	316.9468	136.9323	168.5241	255.0969	88.4740	108.9417	165.0740
	0.5	226.2517	278.2432	420.5294	182.0509	223.9416	338.6279	117.7422	144.9074	219.3350
0.5	-0.5	182.5543	224.6731	340.1077	146.7808	180.6932	273.6735	94.7908	116.7519	177.0119
	0	238.9910	293.9796	444.5344	192.2625	236.5597	357.8888	124.2967	153.0118	231.7219
	0.5	294.6848	362.3669	547.5585	237.1428	291.6818	440.9673	153.4090	188.7842	285.6863
		II								
-0.5	0	320.5896	396.0736	604.2327	258.5407	319.5026	487.6870	168.0158	207.7465	317.4519
	0.5	518.7599	640.4797	975.7674	418.9505	517.3948	788.6859	273.0372	337.3823	514.8584
0	-0.5	353.6125	436.9945	667.0412	285.0143	352.3171	538.0821	185.0137	228.8277	349.8644
	0	554.0891	684.2442	1042.8968	447.2904	552.5111	842.5809	291.2554	359.9696	549.5638
	0.5	749.9710	925.8288	1410.1448	605.8503	748.1218	1140.1111	395.0698	488.1157	744.7025
0.5	-0.5	588.7555	727.1914	1108.7847	475.0936	586.9657	895.4699	309.1225	382.1235	583.6097
	0	785.9046	970.3383	1478.4057	634.6789	783.8408	1194.9221	413.6072	511.0968	780.0073
	0.5	980.9251	1210.8610	1844.0405	792.5426	978.5928	1491.1470	516.9671	638.6823	974.2927
		III								
-0.5	0	665.9289	823.4556	1258.3642	537.4939	664.8104	1016.4554	349.9016	433.0063	662.7254
	0.5	1085.3489	1340.9218	2045.5360	877.1896	1084.0273	1654.5102	572.5646	707.9408	1081.6283
0	-0.5	732.6979	906.3423	1386.0218	591.0736	731.3426	1118.9821	384.3743	475.8362	728.8006
	0	1157.2027	1430.0792	2182.7231	934.8872	1155.6397	1764.7629	609.7345	754.1007	1152.7752
	0.5	1571.5969	1941.3426	2960.4536	1270.5184	1569.8382	2395.1736	829.7425	1025.7548	1566.6738
0.5	-0.5	1227.5902	1517.4288	2317.1646	991.3968	1225.7867	1872.7885	646.1254	799.2991	1222.4586
	0	1644.7830	2032.1438	3100.1411	1329.2941	1642.7813	2507.4519	867.6174	1072.7855	1639.1483
	0.5	2057.2836	2541.0701	3874.3140	1663.3938	2055.0889	3134.9822	1086.6250	1343.2034	2051.1612

Table 2.9
Values of frequency parameter Ω for C-F plate for ε = 0.1

		η								
		-0.5			0			1		
		μ			μ			μ		
α	β	-0.5	0	1	-0.5	0	1	-0.5	0	1
		I								
-0.5	0	4.4740	4.9096	5.9133	3.6372	3.9956	4.8221	2.3870	2.6271	3.1818
	0.5	4.2385	4.6957	5.8364	3.4381	3.8120	4.7451	2.2471	2.4950	3.1140
0	-0.5	5.0587	5.5367	6.6384	4.1233	4.5176	5.4270	2.7189	2.9845	3.5977
	0	4.7517	5.2481	6.4780	3.8612	4.2680	5.2761	2.5318	2.8025	3.4736
	0.5	4.7823	5.3582	6.9148	3.8815	4.3519	5.6223	2.5398	2.8508	3.6902
0.5	-0.5	5.2636	5.8003	7.1225	4.2841	4.7246	5.8103	2.8172	3.1114	3.8364
	0	5.2474	5.8600	7.5012	4.2642	4.7652	6.1066	2.7963	3.1285	4.0169
	0.5	5.4239	6.1642	8.3321	4.4039	5.0076	6.7722	2.8835	3.2817	4.4423
		II								
-0.5	0	20.8850	23.6121	29.9903	17.6932	20.0468	25.5717	12.6127	14.3517	18.4642
	0.5	23.5886	26.5172	33.3259	20.0990	22.6396	28.5662	14.4891	16.3855	20.8387
0	-0.5	23.8086	26.9192	34.1682	20.1710	22.8581	29.1448	14.3798	16.3685	21.0580
	0	26.2777	29.5402	37.0989	22.3883	25.2201	31.8036	16.1373	18.2532	23.2063
	0.5	28.6157	32.0613	39.9720	24.4584	27.4584	34.3764	17.7388	19.9934	25.2351
0.5	-0.5	28.9969	32.5974	40.9159	24.7023	27.8290	35.0784	17.8017	20.1400	25.6001
	0	31.2233	34.9775	43.5687	26.6842	29.9541	37.4702	19.3496	21.8090	27.5090
	0.5	33.3231	37.2197	45.9661	28.5431	31.9476	39.6425	20.7871	23.3615	29.2515
		III								
-0.5	0	55.0963	63.0954	82.3044	47.2848	54.2750	71.1307	34.6198	39.9222	52.8153
	0.5	67.6794	77.0849	99.5098	58.5691	66.8639	86.7129	43.6098	50.0223	65.4855
0	-0.5	62.3792	71.5014	93.4262	53.4955	61.4633	80.6963	39.1043	45.1406	59.8384
	0	74.8482	85.3320	110.3506	64.7084	73.9469	96.0790	48.0790	55.2074	72.4230
	0.5	85.9532	97.7177	125.7737	74.6824	85.1030	110.0293	56.0586	64.1878	83.7665
0.5	-0.5	82.0034	93.5671	121.1861	70.8322	81.0150	105.4344	52.5315	60.3751	79.3435
	0	93.1199	105.9515	136.5717	80.8312	92.1878	119.3768	60.5502	69.3928	90.7163
	0.5	103.5045	117.5812	151.2975	90.1640	102.6623	132.6584	68.0329	77.8338	101.4822

Table 2.10
Values of frequency parameter Ω for C-F plate for ε = 0.3

		η								
		-0.5			0			1		
		μ			μ			μ		
α	β	-0.5	0	1	-0.5	0	1	-0.5	0	1
						I				
-0.5	0	6.3679	7.1862	9.1377	5.1283	5.7908	7.3717	3.3114	3.7434	4.7755
	0.5	6.3069	7.1384	9.1759	5.0729	5.7443	7.3904	3.2683	3.7040	4.7730
0	-0.5	7.5284	8.4824	10.7501	6.0715	6.8452	8.6853	3.9310	4.4370	5.6419
	0	7.3211	8.2692	10.5766	5.8939	6.6604	8.5265	3.8036	4.3019	5.5162
	0.5	7.5190	8.5316	11.0521	6.0497	6.8672	8.9031	3.8998	4.4301	5.7518
0.5	-0.5	8.3602	9.4286	12.0150	6.7360	7.6004	9.6943	4.3536	4.9167	6.2817
	0	8.4694	9.5890	12.3565	6.8183	7.7229	9.9600	4.4001	4.9876	6.4419
	0.5	8.7765	9.9821	13.0148	7.0629	8.0361	10.4858	4.5546	5.1859	6.7761
						II				
-0.5	0	32.1621	37.3807	50.3031	26.7633	31.1470	42.0253	18.4563	21.5357	29.2102
	0.5	38.9445	45.0632	60.1385	32.5501	37.7109	50.4485	22.6414	26.2958	35.3481
0	-0.5	37.3159	43.4068	58.4901	31.0322	36.1468	48.8414	21.3732	24.9634	33.9147
	0	44.0029	50.9530	68.0742	36.7545	42.6142	57.0765	25.5343	29.6805	39.9526
	0.5	50.3005	58.1096	77.3162	42.1147	48.7119	64.9641	29.3946	34.0810	45.6647
0.5	-0.5	49.0950	56.8871	76.0830	40.9838	47.5510	63.7613	28.4399	33.0832	44.5905
	0	55.3906	64.0251	85.2544	46.3513	53.6434	71.6032	32.3175	37.4943	50.2893
	0.5	61.4448	70.9165	94.1944	51.4989	59.5079	79.2214	36.0177	41.7175	55.7911
						III				
-0.5	0	85.6673	100.5458	138.0319	72.0221	84.6509	116.5453	50.7219	59.7858	82.7881
	0.5	112.0779	131.0039	178.3613	94.7960	110.9653	151.5210	67.5763	79.3358	108.9750
0	-0.5	97.9669	115.0964	158.3151	82.2690	96.7923	133.5253	57.8037	68.2032	94.6360
	0	124.7385	145.9411	199.0713	105.3744	123.4661	168.9110	74.9283	88.0520	121.1807
	0.5	149.2424	174.2115	236.5409	126.5156	147.9017	201.4156	90.5972	106.2304	145.5423
0.5	-0.5	137.3316	160.8073	219.7064	115.8894	135.8991	186.2261	82.2269	96.7095	133.3173
	0	162.1695	189.4485	257.6261	137.3273	160.6664	219.1391	98.1254	115.1474	158.0067
	0.5	185.7701	216.6802	293.7314	157.6965	184.2118	250.4677	113.2356	132.6782	181.5034

Table 2.11

Values of frequency parameter Ω for C-F plate for ε = 0.5

		η								
		-0.5			0			1		
		μ			μ			μ		
α	β	-0.5	0	1	-0.5	0	1	-0.5	0	1
I										
-0.5	0	10.8494	12.6628	17.2232	8.6541	10.1035	13.7499	5.4937	6.4174	8.7426
	0.5	11.9438	13.9321	18.9623	9.5219	11.1095	15.1269	6.0387	7.0483	9.6047
0	-0.5	13.1438	15.3372	20.8457	10.4907	12.2453	16.6532	6.6676	7.7873	10.6025
	0	14.0042	16.3273	22.1885	11.1686	13.0243	17.7076	7.0876	8.2689	11.2516
	0.5	15.3582	17.9162	24.4078	12.2452	14.2875	19.4718	7.7671	9.0659	12.3643
0.5	-0.5	16.1238	18.7929	25.5145	12.8632	14.9963	20.3695	8.1683	9.5272	12.9522
	0	17.3482	20.2259	27.5087	13.8350	16.1333	21.9513	8.7793	10.2418	13.9457
	0.5	18.8032	21.9396	29.9169	14.9927	17.4969	23.8675	9.5109	11.1034	15.1564
II										
-0.5	0	58.0524	69.3651	98.8334	47.4405	56.7240	80.9326	31.6144	37.8524	54.1542
	0.5	77.6192	92.4712	130.9857	63.6057	75.8262	107.5479	42.6180	50.8728	72.3428
0	-0.5	67.3097	80.4959	114.8827	54.9651	65.7788	94.0101	36.5756	43.8320	62.8188
	0	87.0329	103.7642	147.1940	71.2740	85.0328	120.7835	47.6958	56.9791	81.1501
	0.5	105.9773	126.1499	178.3925	86.9135	103.5248	146.5860	58.3272	69.5652	98.7536
0.5	-0.5	96.4173	115.0288	163.3814	78.9140	94.2111	133.9947	52.7490	63.0596	89.9312
	0	115.5140	137.5828	194.7787	94.6871	112.8516	159.9757	63.4811	75.7591	107.6749
	0.5	134.1809	159.6462	225.5475	110.0930	131.0717	185.4137	73.9480	88.1529	125.0182
III										
-0.5	0	156.0734	187.8198	271.5179	128.4936	154.7432	224.0309	86.9333	104.8462	152.2413
	0.5	222.8635	267.5458	384.8813	184.1402	221.2246	318.7246	125.4815	150.9803	218.1826
0	-0.5	177.6417	213.9461	309.7804	146.0998	176.0865	255.3381	98.6394	119.0589	173.1547
	0	245.5679	295.0015	424.9537	202.7064	243.6940	351.5726	137.8683	165.9950	240.2033
	0.5	310.1313	372.0848	534.6079	256.4989	307.9721	443.1647	175.1374	210.6048	303.9868
0.5	-0.5	267.9911	322.1291	464.5820	221.0341	265.8844	384.0407	150.0848	180.8094	261.9487
	0	333.3273	400.1218	575.4899	275.4763	330.9284	476.6929	187.8102	225.9592	326.4852
	0.5	396.9417	476.0778	683.5581	328.4794	394.2673	566.9603	224.5355	269.9204	389.3489

Table 2.12
Values of frequency parameter Ω for C-F plate for ε = 0.7

		η								
		-0.5			0			1		
		μ			μ			μ		
α	β	-0.5	0	1	-0.5	0	1	-0.5	0	1
					I					
-0.5	0	25.8231	31.2649	45.7958	20.4024	24.7044	36.1930	12.7254	15.4115	22.5868
	0.5	34.0336	41.1562	60.1568	26.8854	32.5145	47.5327	16.7641	20.2771	29.6515
0	-0.5	30.7781	37.2776	54.6378	24.3216	29.4608	43.1896	15.1749	18.3852	26.9636
	0	38.6661	46.7695	68.3881	30.5476	36.9526	54.0423	19.0508	23.0490	33.7191
	0.5	47.1272	56.9747	83.2397	37.2295	45.0122	65.7719	23.2147	28.0717	41.0298
0.5	-0.5	43.3904	52.4961	76.7922	34.2829	41.4811	60.6897	21.3839	25.8782	37.8741
	0	51.6905	62.5016	91.3370	40.8367	49.3816	72.1749	25.4668	30.8003	45.0300
	0.5	60.2401	72.8180	106.3627	47.5889	57.5294	84.0429	29.6749	35.8785	52.4279
					II					
-0.5	0	146.2937	179.8130	271.4470	117.3383	144.2589	217.8822	75.4280	92.7791	140.2679
	0.5	223.6442	274.5302	413.3528	179.5904	220.5064	332.1714	115.7152	142.1475	214.3383
0	-0.5	165.3364	203.3239	307.2567	132.5507	163.0461	246.5135	85.1288	104.7663	158.5571
	0	243.2413	298.7045	450.1078	195.2604	239.8420	361.5867	125.7266	154.5079	233.1643
	0.5	319.8619	392.5369	590.7177	256.9168	315.3667	474.8153	165.6179	203.3952	306.5262
0.5	-0.5	262.6821	322.6920	486.5972	210.8013	259.0230	390.7816	135.6501	166.7629	251.8389
	0	339.6458	416.9372	627.8022	272.7396	334.8873	504.5009	175.7311	215.8791	325.5333
	0.5	415.9806	510.4228	767.9026	334.1629	410.1290	617.3131	215.4679	264.5784	398.6178
					III					
-0.5	0	398.4390	491.8910	749.2096	321.0219	396.4209	604.1182	208.2538	257.3034	392.5310
	0.5	637.7497	786.5533	1195.6321	514.5929	634.8323	965.5258	334.8164	413.2736	629.2384
0	-0.5	442.2663	546.2304	832.6877	356.1221	439.9519	671.0290	230.7492	285.2172	435.4841
	0	683.8788	843.7045	1283.2997	551.5650	680.6514	1035.8498	358.5473	442.6993	674.4523
	0.5	920.7057	1135.3229	1725.1561	743.1232	916.5948	1393.5515	483.7901	597.0498	908.7228
0.5	-0.5	729.3515	900.0541	1369.7737	588.0030	725.8177	1105.2007	381.9253	471.6932	719.0210
	0	967.4837	1193.2687	1814.0142	780.6216	963.0587	1464.8425	507.8666	626.8993	954.5737
	0.5	1203.3650	1483.7280	2254.1304	971.4138	1198.0626	1821.1323	632.6075	780.6336	1187.9164

Table 2.13

Comparison of frequency parameter Ω for homogeneous ($\mu = 0.0$, $\eta = 0.0$) annular plate of uniform thickness ($\alpha = 0.0$, $\beta = 0.0$)

Boundary	I		II		III	
$\varepsilon = 0.3$						
C-C	45.3462 45.2° 45.346†	45.3371* 45.3462°°	125.3621 125° 125.36†	125.6191* 125.3621°°	246.1573 246.17†	246.6994* 246.1563°°
C-S	29.9777 29.9°	29.9689* 29.9777°°	100.4228 100°	100.6065* 100.4228°°	211.1294	211.5629* 211.1291°°
C-F	6.6604 6.66°	6.6542* 6.6604°°	42.6142 42.6°	42.6156* 42.6141°°	123.4661	123.5739* 123.4662°°
$\varepsilon = 0.5$						
C-C	89.2508 89.2° 89.251†	89.2962* 89.2508°°	246.3428 246° 246.35†	247.0133* 246.3428°°	483.2237 483.25†	484.4110* 483.2216°°
C-S	59.8199 59.8°	59.8468* 59.8200°°	198.0535 198°	198.5584* 198.0535°°	415.1567	416.1242* 415.1563°°
C-F	13.0243 13.0°	13.0206* 13.0243°°	85.0328 85.1°	85.0943* 85.0328°°	243.6940	243.9699* 243.6940°°

* Values taken from Verma[1987].
° Values taken from Leissa[1969].
°° Values taken from Sharma[2006].
† Values taken from Selmane and Lakis[1999].

Table 2.14

Comparison of frequency parameter Ω for homogeneous (μ = 0.0, η = 0.0) annular plate of linear thickness variation (β = 0.0)

α	C-C		C-S	
	I	II	I	II
		ε = 0.3		
-0.5	29.7276 29.7277* 29.720†	82.6289 82.6288* 82.7968†	21.2689 21.2689* 21.2638†	67.7557 67.7558* 67.8800†
-0.3	36.1158 36.1163* 36.120† 36.11578°	100.1368 100.1369* 100.13680°	24.8562 24.8561* 24.858† 24.85620°	81.1247 81.1248* 81.12467°
-0.1	42.3023 42.3017* 42.300†	117.0511 117.0509*	28.2948 28.2880* 28.291†	94.0612 94.0613*
0.1	48.3654 48.3557* 48.347†	133.6001 133.603*	31.6425 31.6579* 31.627†	106.731 106.7312*
0.3	54.3459 54.2882* 54.306† 54.3459°	149.9038 149.8837* 149.90380°	34.9294 34.9122* 34.898† 34.92943°	119.2213 119.2282* 119.22130°
0.5	60.2677 60.1621* 60.2016†	166.0329 166.0372* 166.2233†	38.1733 38.1483* 38.1228†	131.5838 131.572* 131.6944†
		ε = 0.5		
-0.5	54.8912 54.8913* 54.9098†	151.9449 151.945* 152.3412†	39.0056 39.0056* 39.0185†	124.3055 124.3057* 124.6153†
-0.3	68.7954 68.7955* 68.79539°	190.169 190.1664* 190.16900°	47.4486 47.4486* 47.44856°	154.1461 154.1461* 154.14600°
-0.1	82.4679 82.4678*	227.7214 227.7225*	55.7221 55.722*	183.4936 183.4939*
0.1	96.0081 96.0082*	264.8899 264.8848*	63.8991 63.8951*	212.5587 212.5587*
0.3	109.4649 109.4664* 109.46480°	301.8158 301.8171* 301.81560°	72.015 72.0141* 72.01501°	241.445 241.4413* 241.44480°
0.5	122.8654 122.8699* 122.8823†	338.578 338.5798* 339.3885†	80.0898 80.0936* 80.0928†	270.2104 270.2085* 270.7872†

* Values taken from Lal[1979].
† Values taken from Verma[1987].
° Values taken from Chen[1997].

Table 2.15

Comparison of frequency parameter Ω for homogeneous (μ = 0.0, η = 0.0) annular plate of parabolic thickness variation (α = 0.0)

β	C-C				C-S			
	I		II		I		II	
ε = 0.3								
-0.5	32.6202	32.6201*	92.0144	92.0138*	24.0735	24.0734*	76.4146	76.4145*
-0.3	37.9240	37.9240*	106.0049	106.0050*	26.6013	26.6013*	86.4757	86.4757*
-0.1	42.9245	42.9222*	119.0712	119.0715*	28.8926	28.8922*	95.8872	95.8872*
0.1	47.7271	47.9402*	131.5259	131.5222*	31.0317	30.8515*	104.8693	104.8668*
0.3	52.3893	52.3417*	143.5418	143.5575*	33.0643	33.1102*	113.5437	113.5269*
0.5	56.9460	56.9390*	155.2258	155.2150*	35.0181	35.0332*	121.9854	121.9536*
ε = 0.5								
-0.5	60.6381	60.6396*	169.2995	169.2996*	44.3594	44.3594*	140.0209	140.0208*
-0.3	72.4007	72.4007*	201.0472	201.0476*	50.7836	50.7836*	163.8999	163.9001*
-0.1	83.7093	83.7092*	231.4658	231.4666*	56.8644	56.8645*	186.8278	186.8305*
0.1	94.7352	94.7337*	261.0512	261.0513*	62.7327	62.7334*	209.1583	209.1586*
0.3	105.5690	105.5722*	290.0676	290.0068*	68.4573	68.4548*	231.0805	231.0857*
0.5	116.2646	116.2645*	318.6732	318.6692*	74.0786	74.0800*	252.7074	252.6975*

* Values taken from Lal[1979].

Table 2.16

Number of grid points for convergence of frequency parameter Ω by using zeros of Chebyshev polynomial, Legendre polynomial and equidistant collocation points for C-C plate for $\eta = 0.5$, $\mu = -0.5$

	$\alpha = -0.5,\ \beta = 0.5$			$\alpha = 0.0,\ \beta = 0.5$			$\alpha = 0.5,\ \beta = -0.5$			$\alpha = 0.5,\ \beta = 0.5$		
Grid points / Mode	I	II	III	I	II	III	I	II	III	I	II	III
Chebyshev polynomial	11	14	16	13	13	16	11	12	15	11	12	15
Legendre polynomial	13	15	17	15	16	16	14	14	17	13	14	17
Equidistant	17	19	23	21	22	22	19	20	22	19	22	25
Liew et al.[1997]	13	16	19	15	16	18	15	15	17	15	16	17

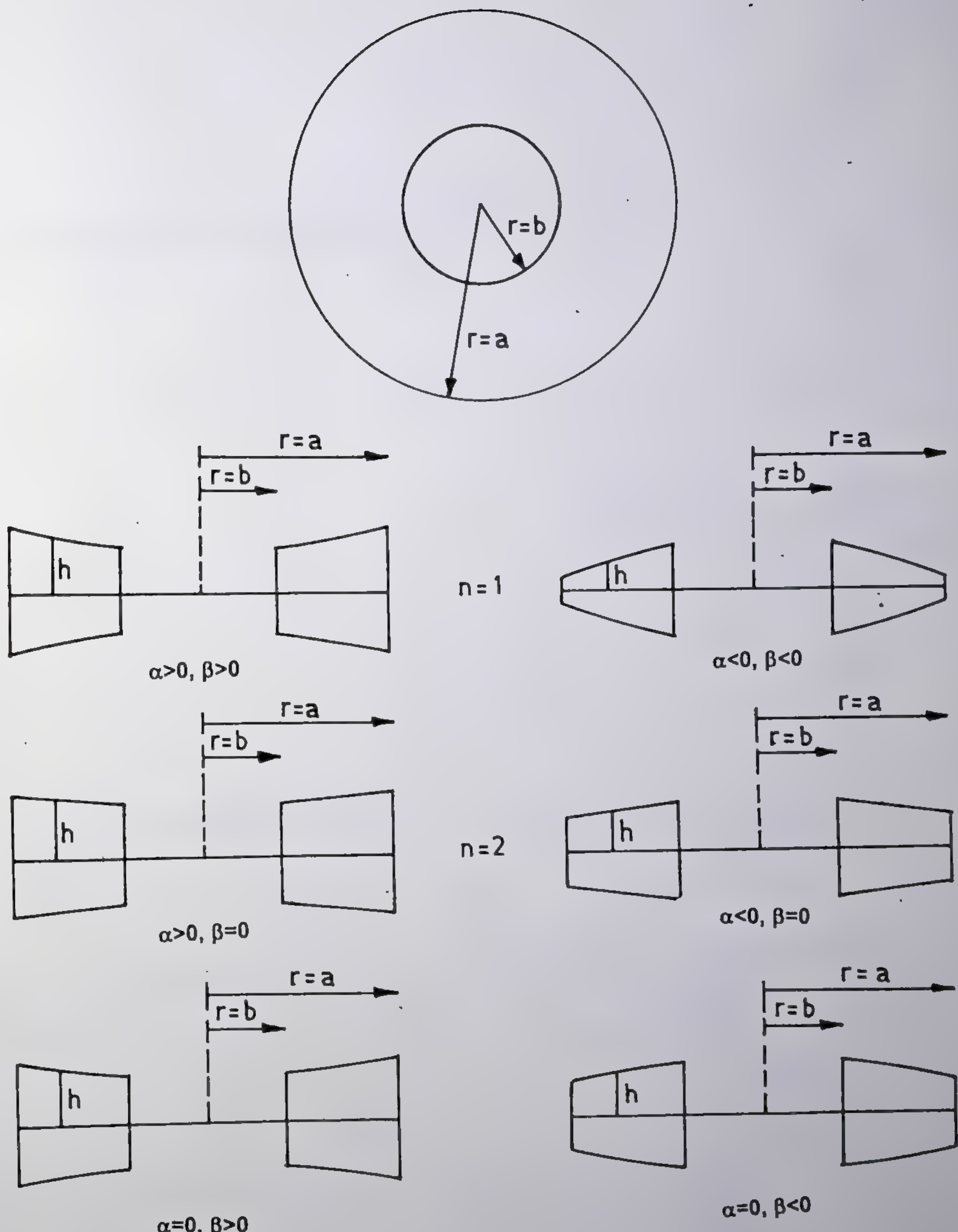

Fig. 2 : Geometry and cross-section of the tapered annular plate for quadratic thickness variation i.e. $\bar{h} = h_0\left(1 + \alpha x + \beta x^2\right)$ where $x = \dfrac{r}{a}$

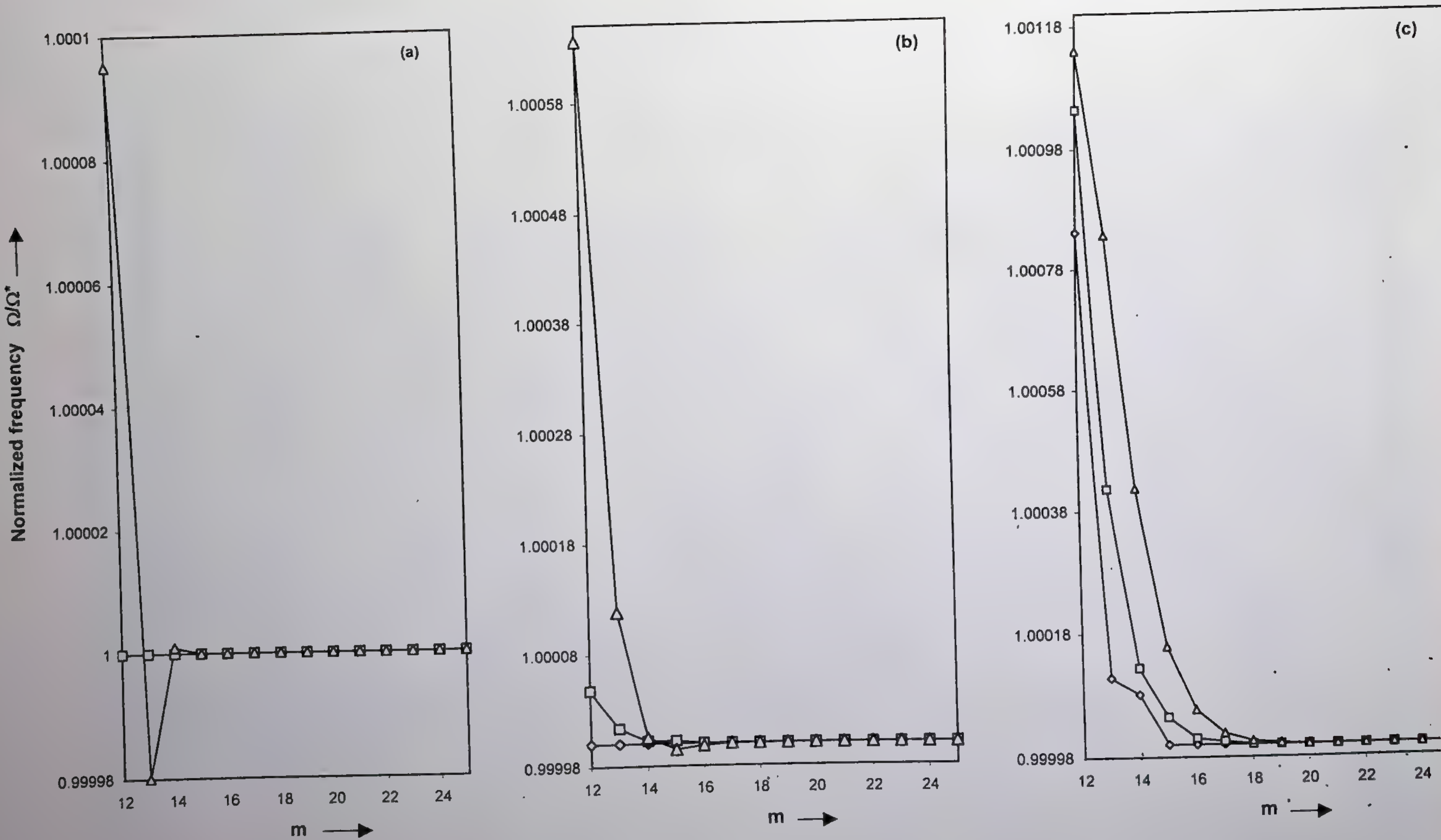

Fig. 2.1 : Convergence of the normalized frequency parameter Ω/Ω^* for the first three modes of vibration with grid refinement for $\eta = 1.0$, $\mu = -0.5$, $\alpha = -0.2$, $\beta = -0.3$, $\varepsilon = 0.3$ for (a) C-C (b) C-S and (c) C-F plate.
--◊--, fundamental mode: --□--, second mode: —Δ—, third mode. Ω^*- the DQ results using 25 grid points.

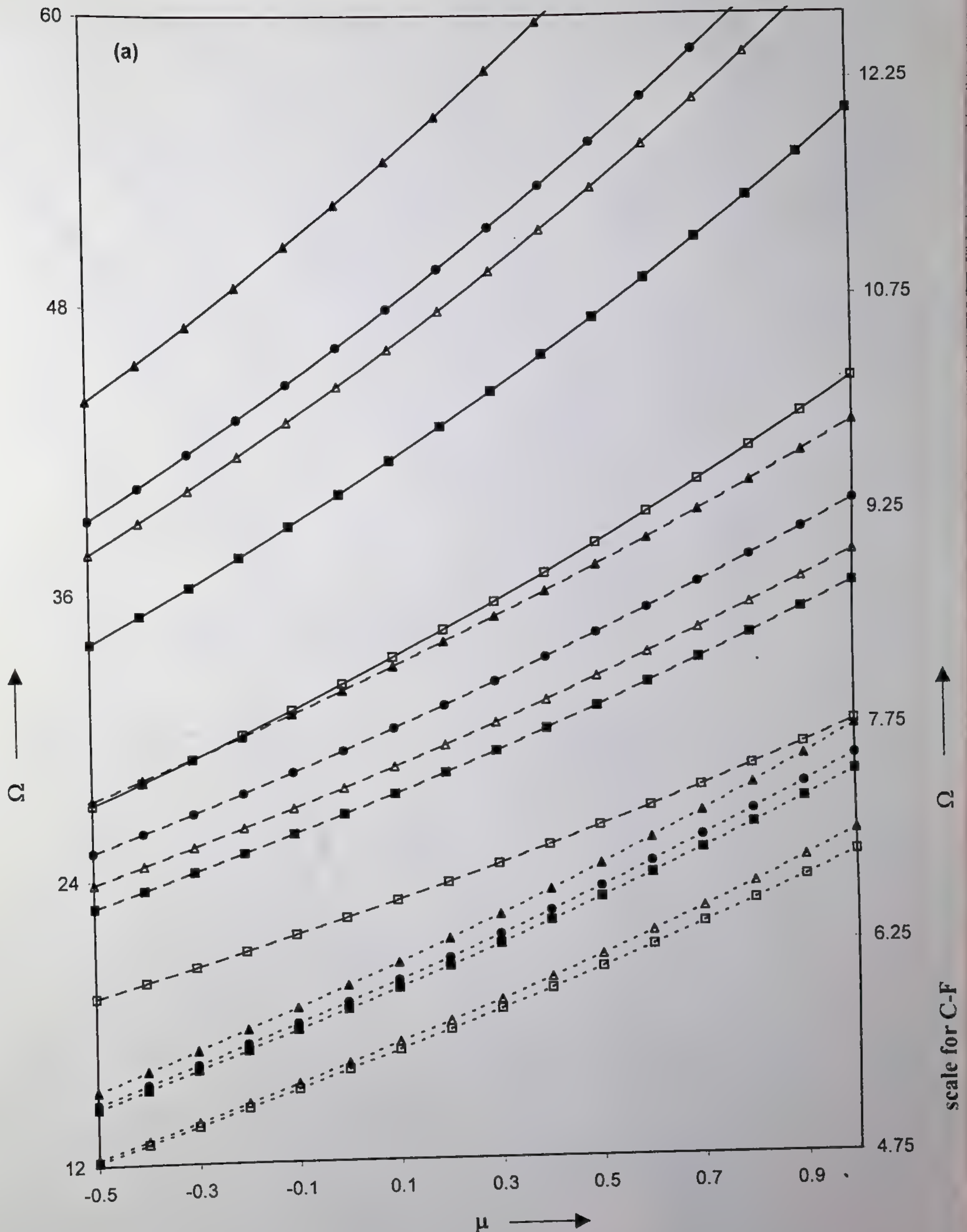

Fig. 2.2a : Frequency parameter for C-C, C-S and C-F plates vibrating in fundamental mode for $\eta = 0.5$, $\varepsilon = 0.3$.———, C-C ; – – – – – –, C-S ; -------------, C-F.
□, $\alpha = 0$, $\beta = -0.3$; Δ, $\alpha = 0$, $\beta = 0.3$; ■, $\alpha = 0.3$, $\beta = -0.3$; ●, $\alpha = 0.3$, $\beta = 0$; ▲, $\alpha = 0.3$, $\beta = 0.3$

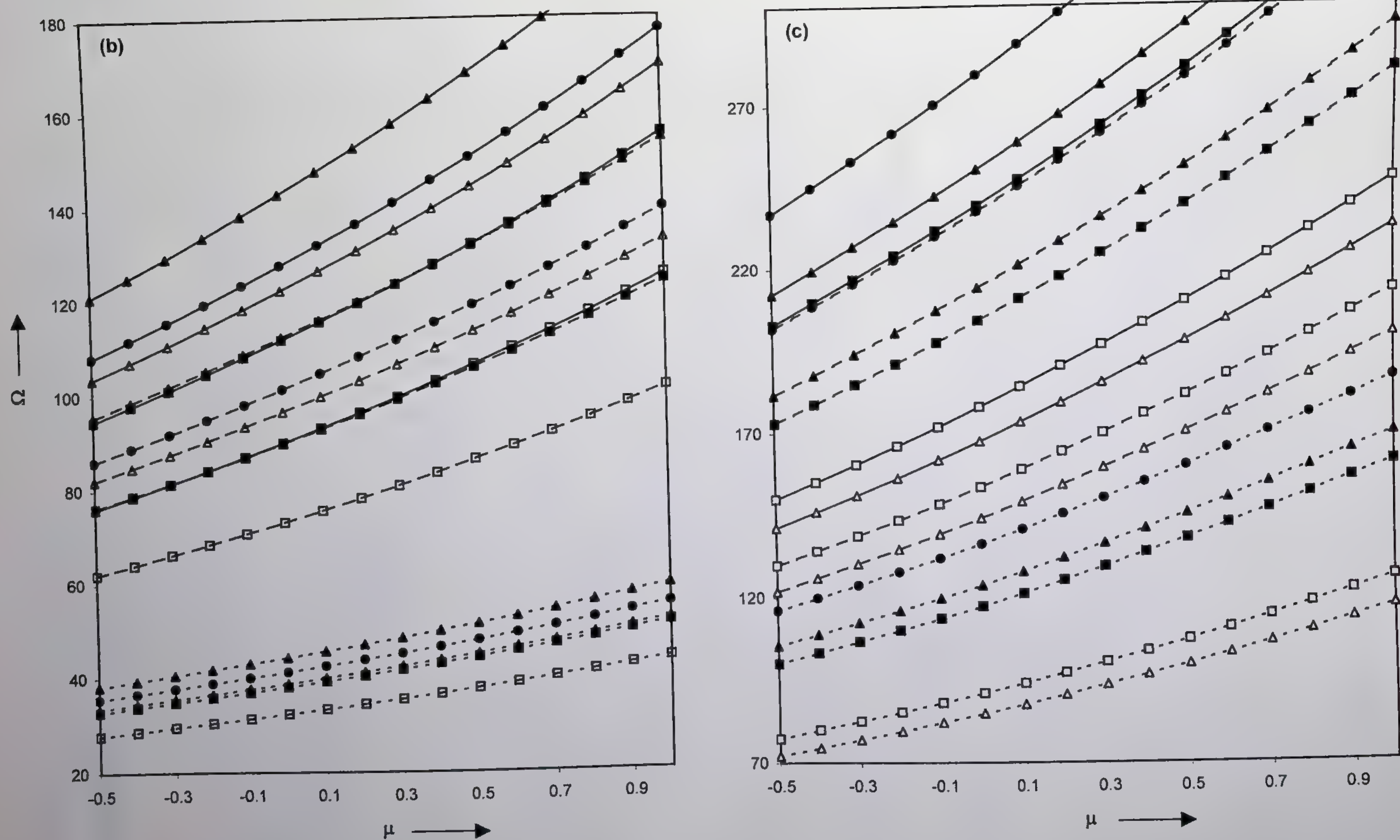

Fig. 2.2 : Frequency parameter for C-C, C-S and C-F plates vibrating in (b) second and (c) third mode for $\eta = 0.5$, $\varepsilon = 0.3$.
———, C-C ; – – – – – –, C-S ; ------------, C-F.
□, $\alpha = 0$, $\beta = -0.3$; Δ, $\alpha = 0$, $\beta = 0.3$; ■, $\alpha = 0.3$, $\beta = -0.3$; ●, $\alpha = 0.3$, $\beta = 0$; ▲, $\alpha = 0.3$, $\beta = 0.3$.

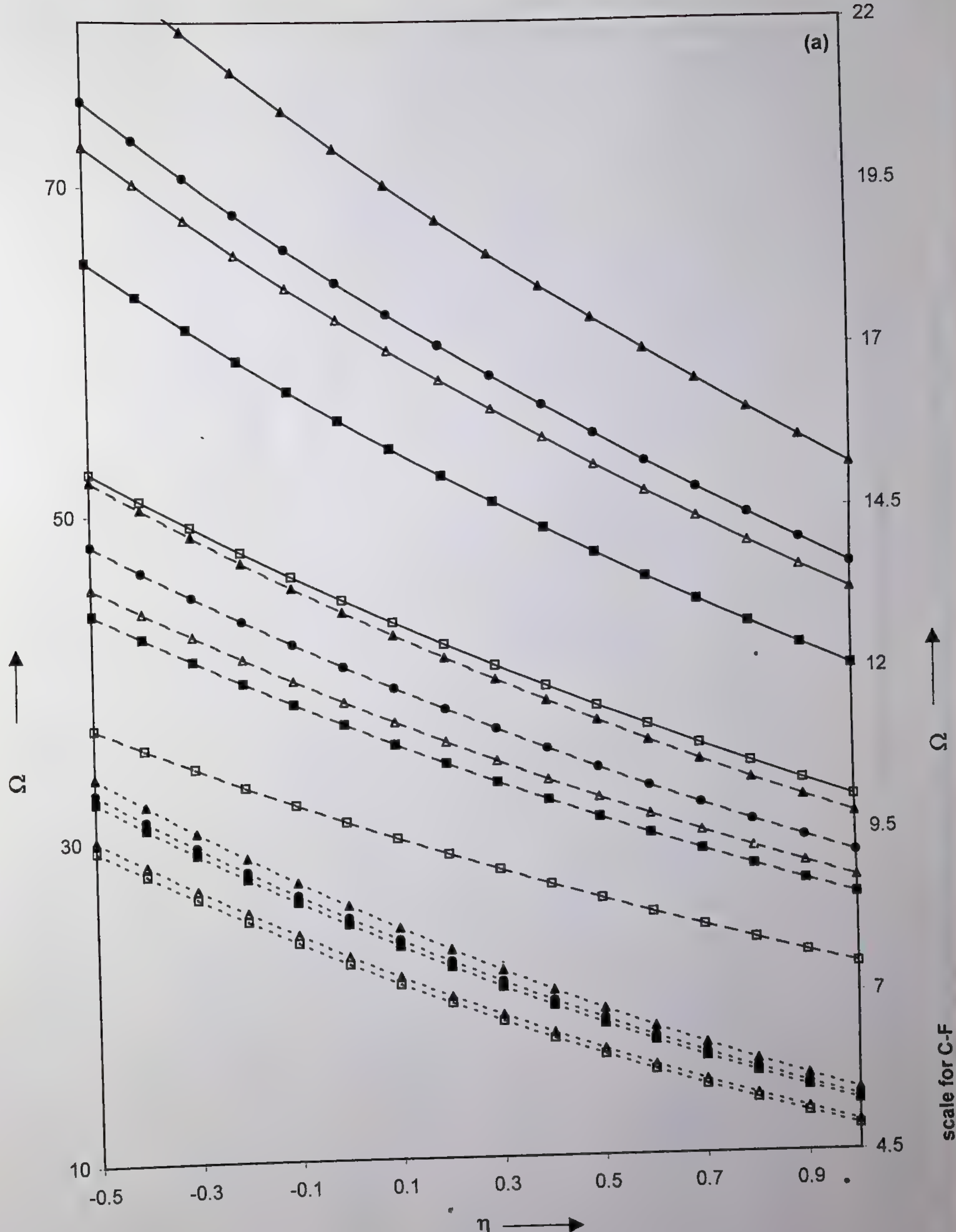

Fig. 2.3a : Frequency parameter for C-C, C-S and C-F plates vibrating in fundamental mode for $\mu = 0.5$, $\varepsilon = 0.3$. ———, C-C ; – – – – – –, C-S ; ------------, C-F.
□, $\alpha = 0$, $\beta = -0.3$; Δ, $\alpha = 0$, $\beta = 0.3$; ■, $\alpha = 0.3$, $\beta = -0.3$; ●, $\alpha = 0.3$, $\beta = 0$; ▲, $\alpha = 0.3$, $\beta = 0.3$

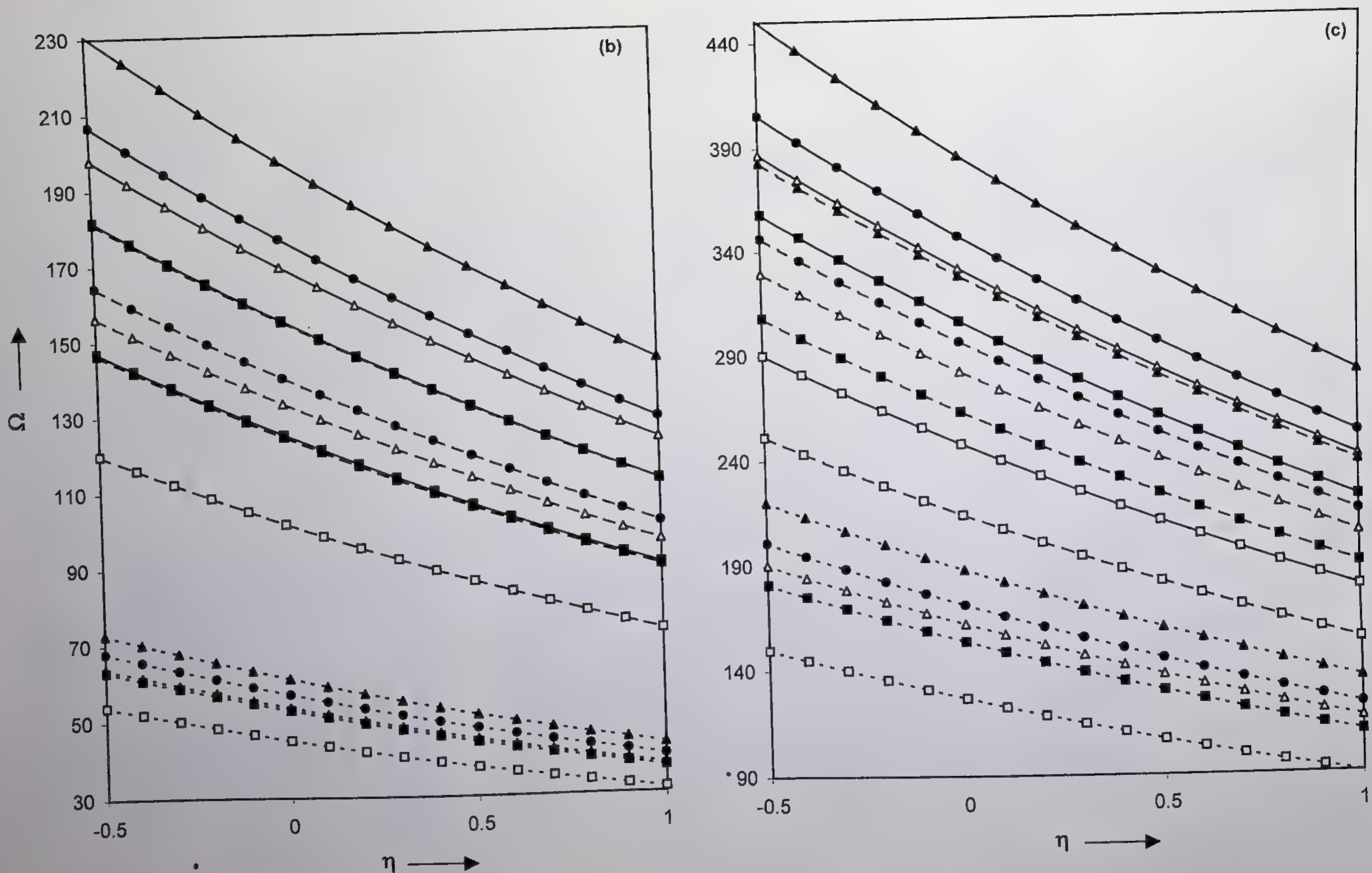

Fig. 2.3 : Frequency parameter for C-C, C-S and C-F plates vibrating in (b) second and (c) third mode for $\mu = 0.5$, $\varepsilon = 0.3$. ———, C-C ; – – – – – –, C-S ; ------------, C-F.
□, $\alpha = 0$, $\beta = -0.3$; Δ, $\alpha = 0$, $\beta = 0.3$; ■, $\alpha = 0.3$, $\beta = -0.3$; ●, $\alpha = 0.3$, $\beta = 0$; ▲, $\alpha = 0.3$, $\beta = 0.3$.

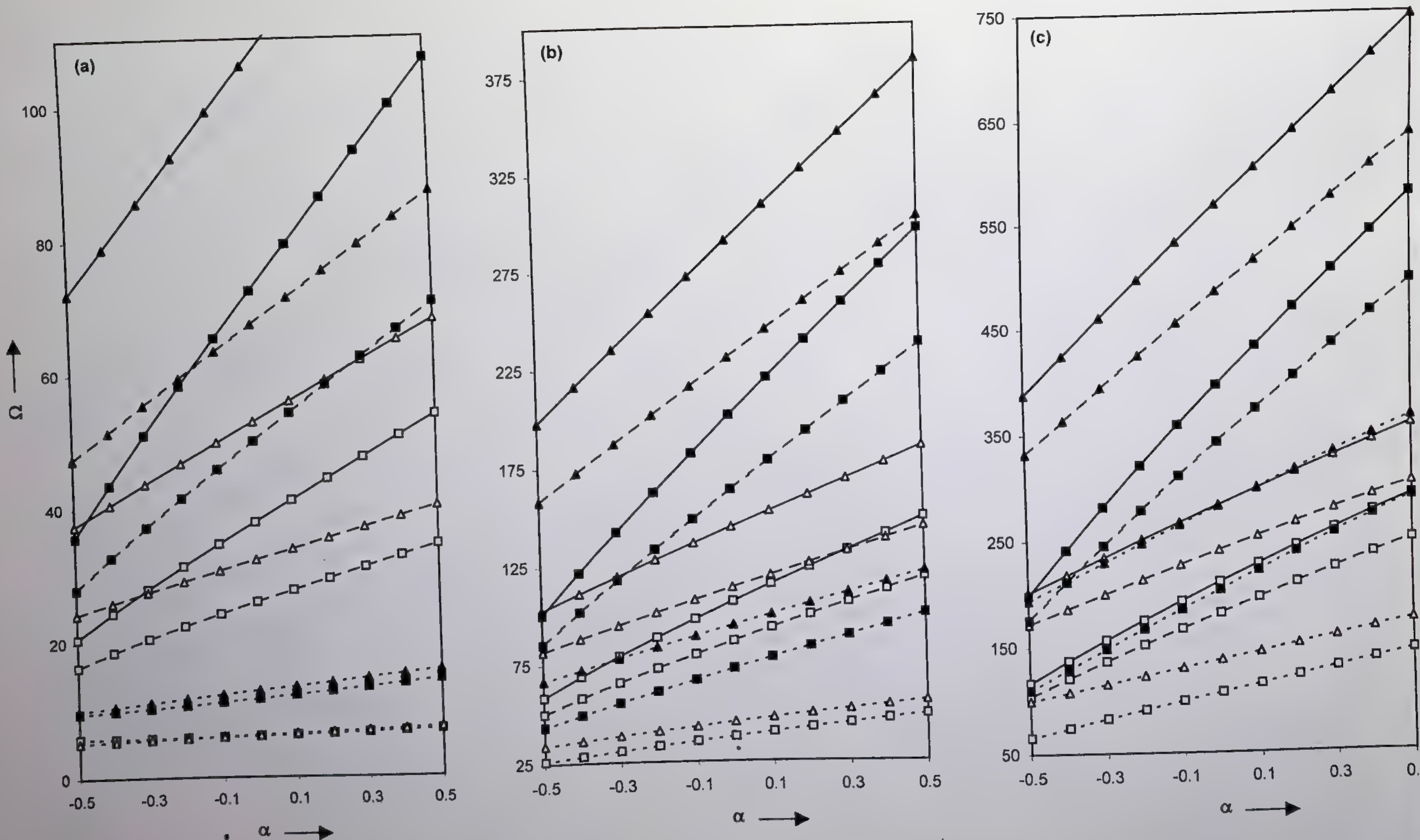

Fig. 2.4 : Frequency parameter for C-C, C-S and C-F plates vibrating in (a) fundamental (b) second and (c) third mode for $\mu = 0.5$, $\eta = 0.5$.
———, C-C ; – – – – – –, C-S ; ------------, C-F.
□, $\varepsilon = 0.3$, $\beta = -0.3$; Δ, $\varepsilon = 0.3$, $\beta = 0.3$; ■, $\varepsilon = 0.5$, $\beta = -0.3$; ▲, $\varepsilon = 0.5$, $\beta = 0.3$.

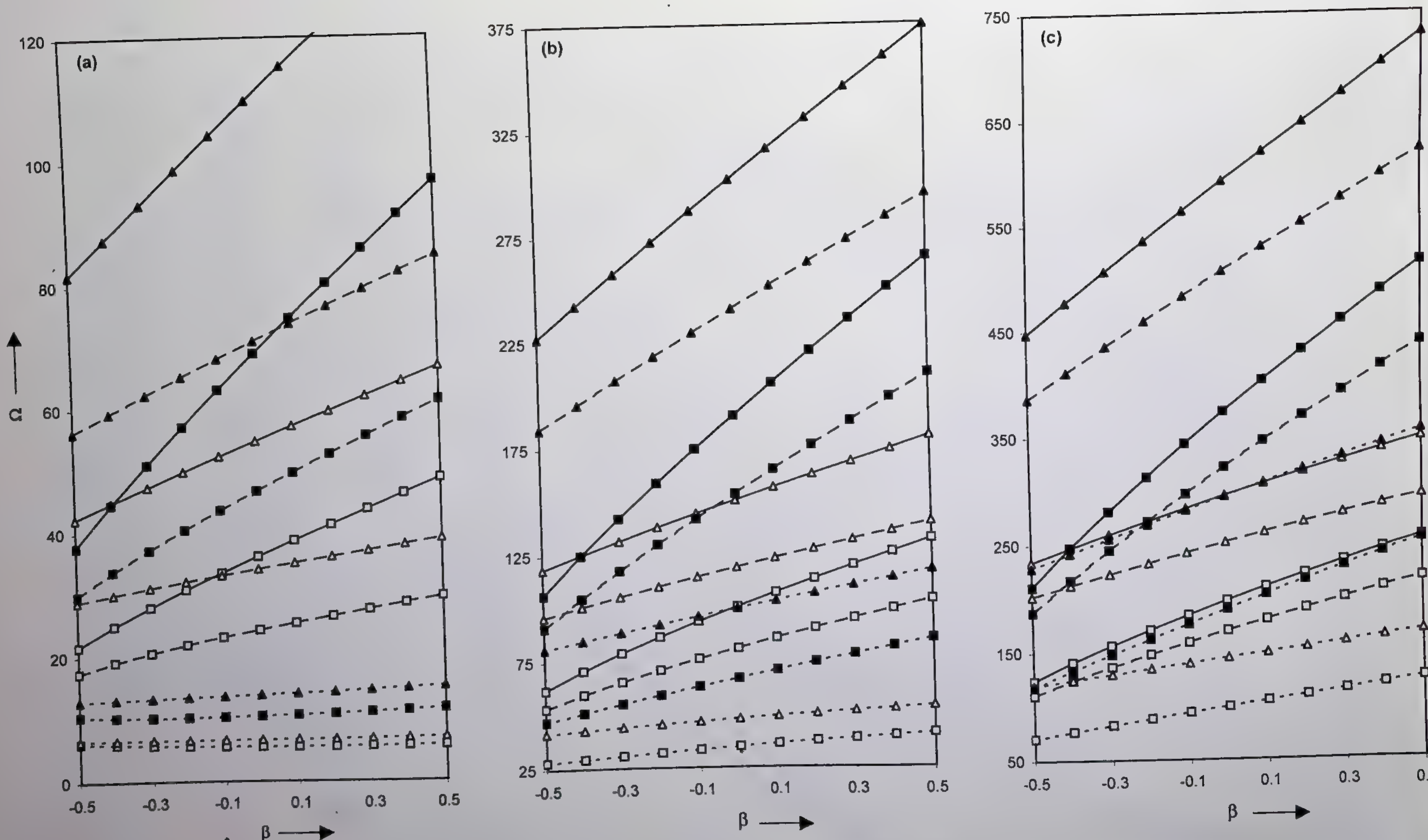

Fig. 2.5 : Frequency parameter for C-C, C-S and C-F plates vibrating in (a) fundamental (b) second and (c) third mode for $\mu = 0.5$, $\eta = 0.5$.

———, C-C ; – – – – –, C-S ; ············, C-F.

□, $\varepsilon = 0.3$, $\alpha = -0.3$; Δ, $\varepsilon = 0.3$, $\alpha = 0.3$; ■, $\varepsilon = 0.5$, $\alpha = -0.3$; ▲, $\varepsilon = 0.5$, $\alpha = 0.3$.

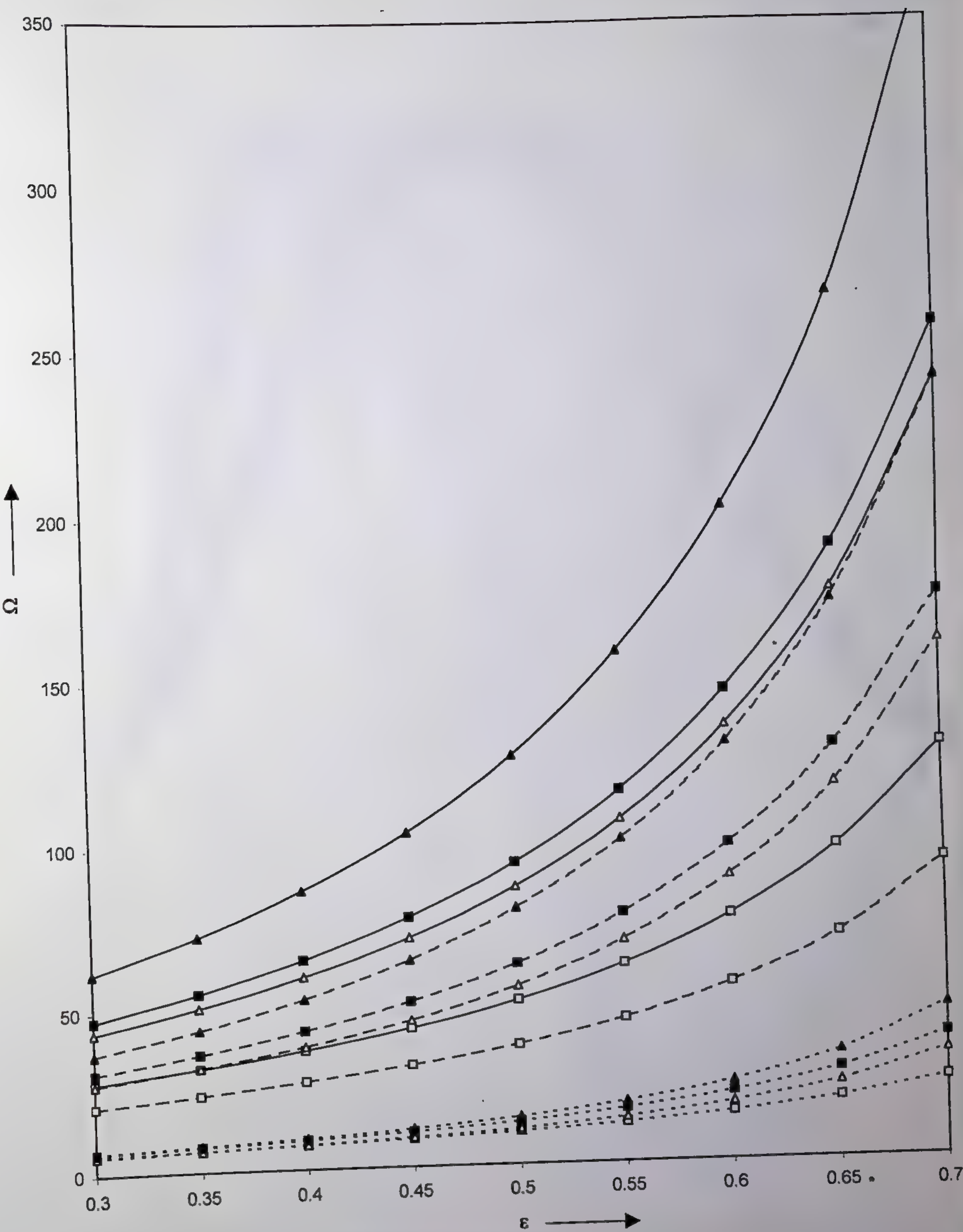

Fig. 2.6 : Frequency parameter for C-C, C-S and C-F plates vibrating in fundamental mode for $\mu = 0.5$, $\eta = 0.5$.———, C-C ; – – – – – –, C-S ; ·············, C-F.
□, $\alpha = -0.3$, $\beta = -0.3$; Δ, $\alpha = -0.3$, $\beta = 0.3$; ■, $\alpha = 0.3$, $\beta = -0.3$; ▲, $\alpha = 0.3$, $\beta = 0.3$.

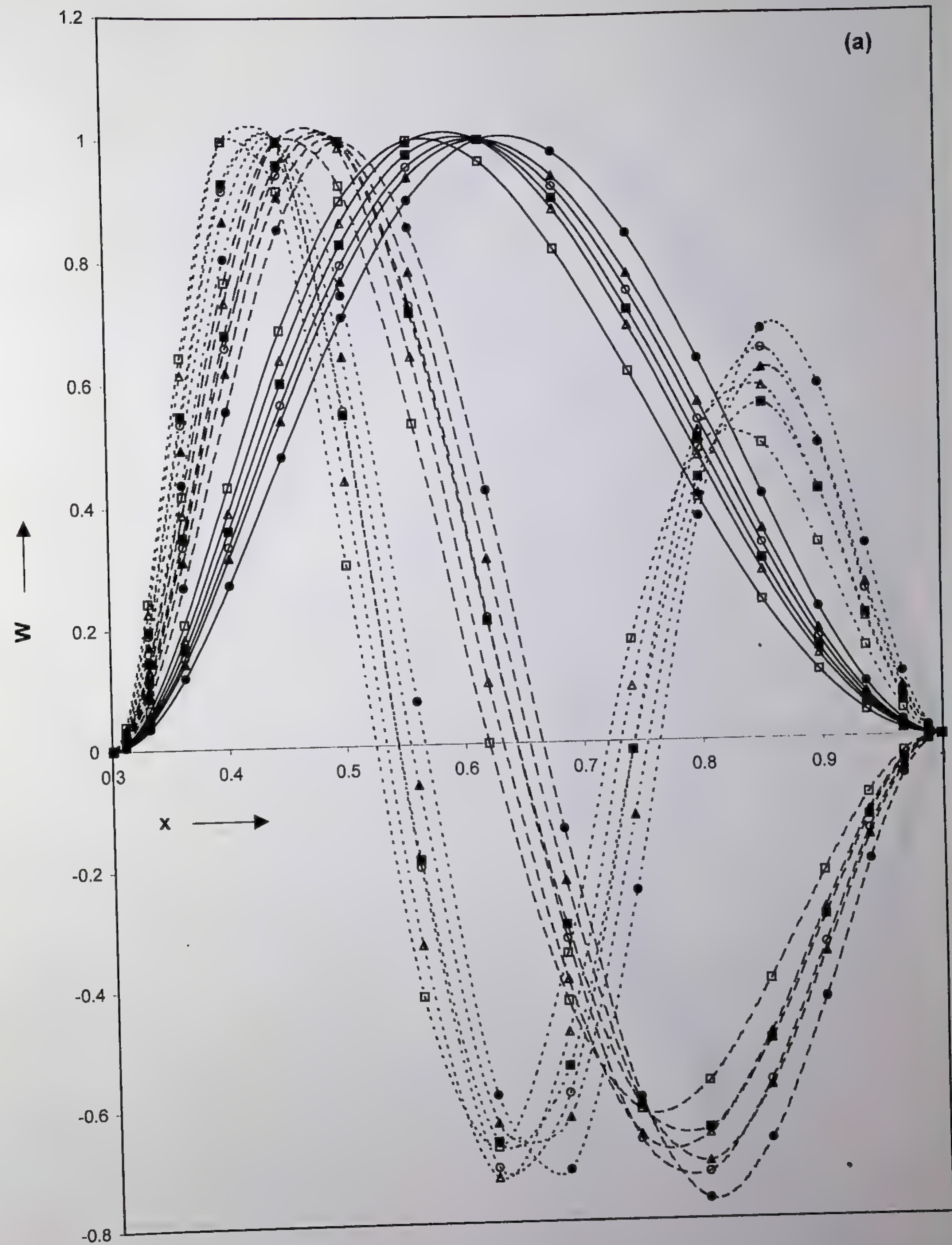

Fig. 2.7a : Normalized displacements for the first three modes of vibration for C-C plate for $\eta = 0.5$, $\varepsilon = 0.3$. ———, fundamental mode; – – – , second mode; ------, third mode. □, $\alpha = 0.5$, $\beta = 0.5$; Δ, $\alpha = 0.5$, $\beta = 0$; ○, $\alpha = 0$, $\beta = 0$. □, Δ, ○, $\mu = 1.0$; ■, ▲, ●, $\mu = -0.5$.

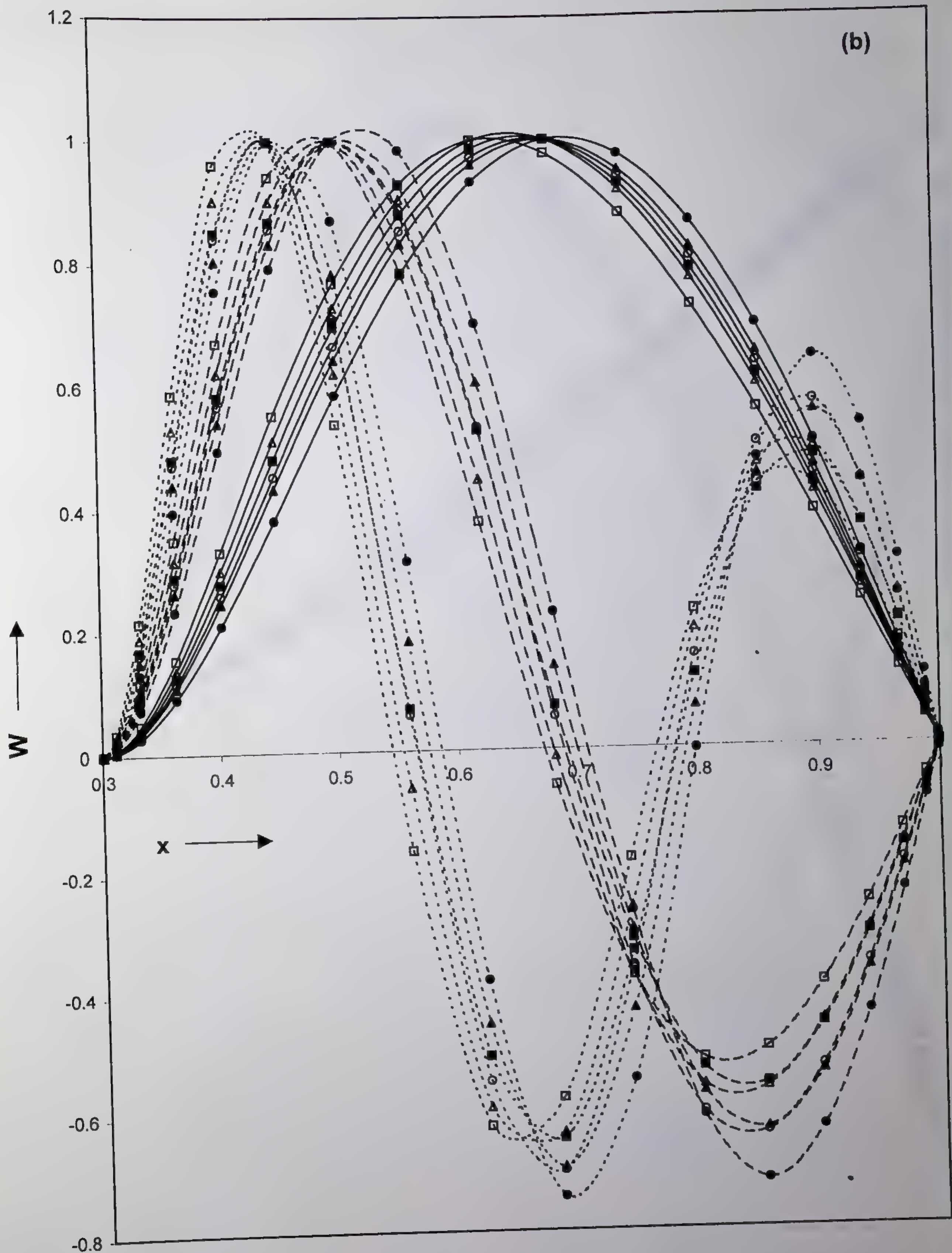

Fig. 2.7b : Normalized displacements for the first three modes of vibration for C-S plate for $\eta = 0.5$, $\varepsilon = 0.3$. ———, fundamental mode; – – – , second mode; ------, third mode. □, $\alpha = 0.5$, $\beta = 0.5$; Δ, $\alpha = 0.5$, $\beta = 0$; ○, $\alpha = 0$, $\beta = 0$. □, Δ, ○, $\mu = 1.0$; ■, ▲, ●, $\mu = -0.5$.

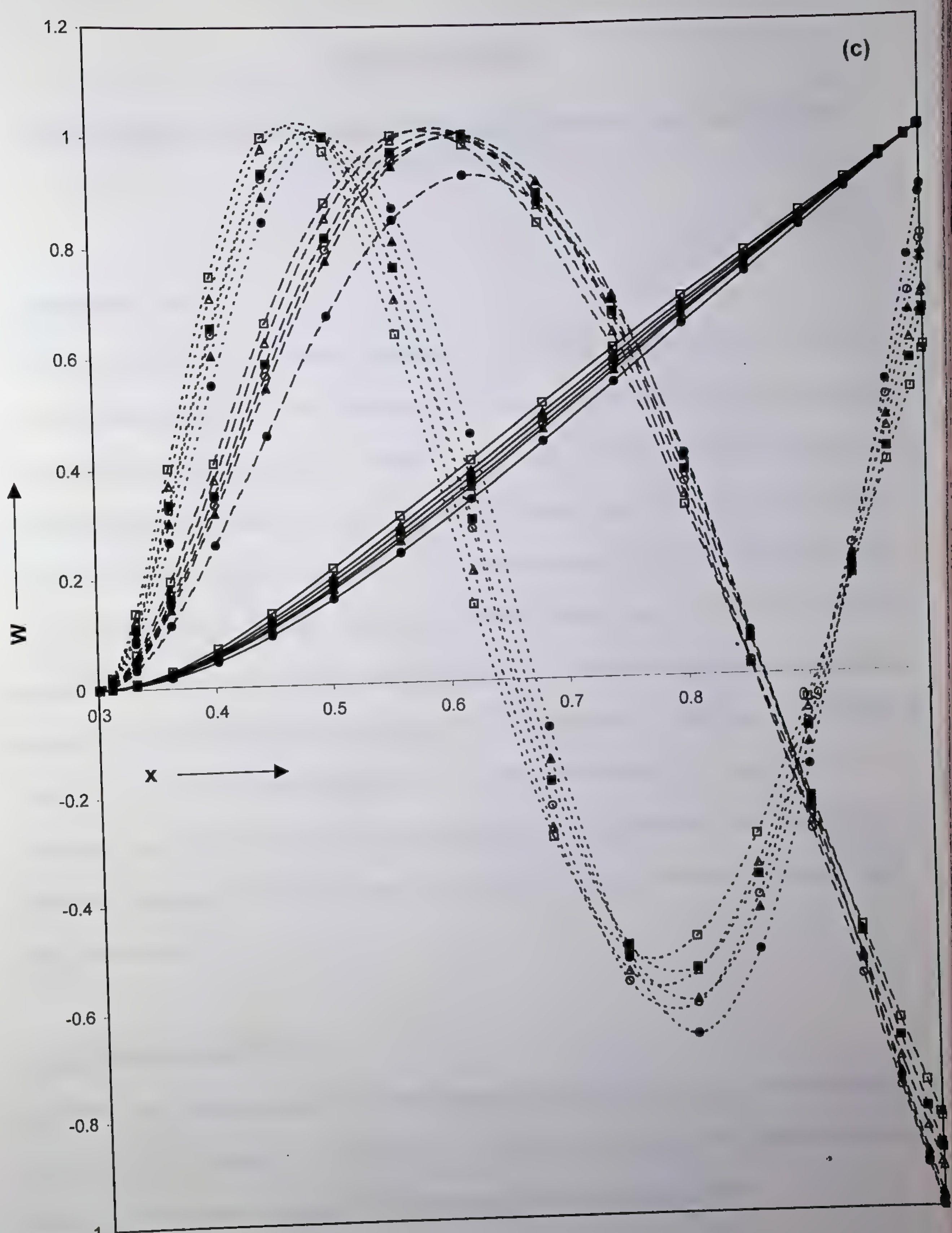

Fig. 2.7c : Normalized displacements for the first three modes of vibration for C-F plate for $\eta = 0.5$, $\varepsilon = 0.3$. ———, fundamental mode; – – –, second mode; -------, third mode. □, $\alpha = 0.5$, $\beta = 0.5$; Δ, $\alpha = 0.5$, $\beta = 0$; ○, $\alpha = 0$, $\beta = 0$. □, Δ, ○, $\mu = 1.0$; ■, ▲, ●, $\mu = -0.5$.

CHAPTER III

VIBRATIONS OF NON-HOMOGENEOUS CIRCULAR PLATES OF QUADRATIC THICKNESS

1. INTRODUCTION

Circular plates of uniform/non-uniform thickness are extensively used as structural components in various engineering fields such as aerospace industry, missile technology, naval ship design and telephone industry etc. Keeping the above in view, a DQ procedure has been developed for obtaining natural frequencies of non-homogeneous circular plates of quadratically varying thickness employing classical plate theory. The non-homogeneity of plate material may arise due to the variation of Young's modulus and density which have been assumed to vary exponentially in radial direction. The numerical solution of the governing differential equation derived by using Hamilton's energy principle has been obtained by differential quadrature method. The effect of non-homogeneity on natural frequencies of vibration has been investigated for different values of density parameter and taper parameters for three boundary conditions. Transverse displacements have been presented for a specified plate for the first three modes of vibration.

2. EQUATION OF MOTION

The differential equation which governs axisymmetric motion of an isotropic non-homogeneous circular plate of radius a with thickness $h = h(r)$ referred to a cylindrical polar coordinate system (r, θ, z) (Figure 3) is given by the equation (2.2.21), which is as follows:

$$
\begin{aligned}
&Eh^3\frac{\partial^4 w}{\partial r^4}+\left[\frac{2}{r}\left\{Eh^3+r\left(h^3\frac{dE}{dr}+3Eh^2\frac{dh}{dr}\right)\right\}\right]\frac{\partial^3 w}{\partial r^3}\\
&+\left[\frac{1}{r^2}\left\{\begin{array}{l}-Eh^3+r(2+\upsilon)\left(h^3\dfrac{dE}{dr}+3Eh^2\dfrac{dh}{dr}\right)\\ +r^2\left(h^3\dfrac{d^2E}{dr^2}+6h^2\dfrac{dE}{dr}\dfrac{dh}{dr}+3E\left(2h\left(\dfrac{dh}{dr}\right)^2+h^2\dfrac{d^2h}{dr^2}\right)\right)\end{array}\right\}\right]\frac{\partial^2 w}{\partial r^2}\\
&+\left[\frac{1}{r^3}\left\{Eh^3-r\left(h^3\frac{dE}{dr}+3Eh^2\frac{dh}{dr}\right)+r^2\upsilon\left(\begin{array}{l}h^3\dfrac{d^2E}{dr^2}+6h^2\dfrac{dE}{dr}\dfrac{dh}{dr}\\ +3E\left(2h\left(\dfrac{dh}{dr}\right)^2+h^2\dfrac{d^2h}{dr^2}\right)\end{array}\right)\right\}\right]\frac{\partial w}{\partial r}\\
&\qquad\qquad+12\rho h\left(1-\upsilon^2\right)\frac{\partial^2 w}{\partial t^2}=0.
\end{aligned}
\qquad (3.2.1)
$$

This can be derived by assuming the relations (2.2.1-2.2.6) and replacing the integration limits with respect to r from 0 to a instead of b to a in equation (2.2.18).

Introducing non-dimensional variables $x=\dfrac{r}{a}$, $\overline{w}=\dfrac{w}{a}$, $\overline{h}=\dfrac{h}{a}$, together with quadratic variation in thickness i.e.

$$\overline{h}=h_0\left(1+\alpha x+\beta x^2\right), \text{ such that } |\alpha|\le 1, |\beta|\le 1 \text{ and } \alpha+\beta>-1, \qquad (3.2.2)$$

and assuming exponential variation for non-homogeneity of material as follows :

$$E=E_0e^{\mu x}, \qquad \rho=\rho_0e^{\eta x} \qquad (3.2.3)$$

equation (3.2.1) now reduces to

$$P_0\frac{d^4W}{dx^4}+P_1\frac{d^3W}{dx^3}+P_2\frac{d^2W}{dx^2}+P_3\frac{dW}{dx}+P_4W=0, \qquad (3.2.4)$$

where, $\overline{w}(x,t)=W(x)e^{i\omega t}$ (for harmonic vibrations), ω is the radian frequency, h_0, ρ_0 are the thickness and density at the centre of the plate, μ and η are non-homogeneity parameters, α

and β are taper parameters, and variable coefficients P_i, i = 0, 1, 2, 3, 4 are given by relations (2.2.25).

An approximate solution of equation (3.2.4) together with boundary conditions at the edge $x = 1$ and regularity condition at the centre $x = 0$, has been obtained by DQ method.

3. METHOD OF SOLUTION : DQM

Let x_i, $i = 1, 2, \ldots, m$ be the grid points in the applicability range [0,1] of the plate. The DQ method (Bert et al.[1988]) approximates the n^{th} order derivative of $W(x)$ with respect to x at discrete point x_i as

$$W_x^{(n)}(x_i) = \sum_{j=1}^{m} c_{ij}^{(n)} W(x_j) \ , \qquad i = 1, 2, \ldots, m \ , \qquad (3.3.1)$$

where weighting coefficients $c_{ij}^{(n)}$ are determined as in chapter II using relations (2.3.2-2.3.5).

Now, discretizing equation (3.2.4) at the grid point $x = x_i$ and substituting the values of first four derivatives of W from equation (3.3.1), we get

$$\sum_{j=1}^{m} (P_0 c_{ij}^{(4)} + P_{1,i} c_{ij}^{(3)} + P_{2,i} c_{ij}^{(2)} + P_{3,i} c_{ij}^{(1)}) W(x_j) + P_{4,i} W(x_i) = 0 \qquad \text{for } i = 2, 3, \ldots, (m-2). \qquad (3.3.2)$$

The satisfaction of equation (3.3.2) at (m-3) grid points x_i . i = 2, 3,...., (m-2) together with the regularity condition at the center provides a set of (m-2) equations in terms of unknowns $W_j (\equiv W(x_j))$, $j = 1,2,\ldots,m$. The resulting system of equations can be written in the matrix form as

$$[B][W^*] = [0] \ , \qquad (3.3.3)$$

where B and W^* are matrices of order (m-2) x m and m x 1, respectively.

The (m-2) internal grid points chosen for collocation are the zeros of shifted Chebyshev polynomial of order (m-2) with orthogonality range (0,1) given by

$$x_{k+1} = \frac{1}{2}\left[1 + \cos\left(\frac{2k-1}{m-2}\frac{\pi}{2}\right)\right], \qquad k = 1, 2, ..., (m\text{-}2) \quad . \tag{3.3.4}$$

However, for a specified plate, a comparative study has been made considering four different sets of grid points, namely (i) zeros of shifted Chebyshev polynomial (ii) zeros of shifted Legendre polynomial (iii) grid points taken by Liew et al.[1997] (iv) equally spaced grid points.

4. BOUNDARY CONDITIONS AND FREQUENCY EQUATIONS

By satisfying the relations,

(i) $W = \frac{dW}{dx} = 0$ for clamped edge,

(ii) $W = \frac{d^2W}{dx^2} + \frac{\upsilon}{x}\frac{dW}{dx} = 0$ for simply–supported edge, and

(iii) $\frac{d^2W}{dx^2} + \frac{\upsilon}{x}\frac{dW}{dx} = \frac{d^3W}{dx^3} + \frac{1}{x}\frac{d^2W}{dx^2} - \frac{1}{x^2}\frac{dW}{dx} = 0$ for free edge,

a set of two homogeneous equations in terms of W_j is obtained. For a clamped plate. these equations together with field equations (3.3.3) give a complete set of m equations in m unknowns, which can be written as

$$\begin{bmatrix} B \\ B^C \end{bmatrix}\left[W^{*}\right] = [0] \quad , \tag{3.4.1}$$

where B^C is a matrix of order 2 x m.

For a non-trivial solution of equation (3.4.1), the frequency determinant must vanish and hence

$$\begin{vmatrix} B \\ B^C \end{vmatrix} = 0 \ . \tag{3.4.2}$$

Similarly, for simply supported and free edge boundary conditions, the frequency determinants can respectively be written as

$$\begin{vmatrix} B \\ B^S \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} B \\ B^F \end{vmatrix} = 0 \ . \tag{3.4.3, 3.4.4}$$

5. NUMERICAL RESULTS AND DISCUSSION

First three natural frequencies of vibration have been computed from equations (3.4.2-3.4.4) for all the three boundary conditions. The values of various plate parameters taken are as follows :

non-homogeneity parameter μ = -0.5(0.1)1.0;

density parameter η = -0.5(0.1)1.0 and

taper constants α = -0.5(0.1)0.5 ; β = -0.5(0.1)0.5 (such that $\alpha+\beta > -1$)

for $\upsilon = 0.3$.

The convergence of the method with the number of grid points m has been carried out as in chapter II for different sets of plate parameters for all the three boundary conditions. In all the computations, the number of grid points has been taken as m = 18, since further increase in m does not improve the results even in the fourth place of decimal (Figs. 3.1(a, b, c)).

The numerical results are given in Tables (3.1-3.9) and Figures (3.2-3.6). Tables (3.1-3.9) give the value of frequency parameter Ω for different values of plate parameters i.e. η = -0.5, 0.0,

1.0, μ = -0.5, 0.0, 1.0, α = -0.5, -0.1, 0.0, 0.1, 0.5; β = -0.5, -0.1, 0.0, 0.1, 0.5 (such that $\alpha + \beta > -1$) for clamped, simply supported and free plates, respectively. From the results, it is found that for $\alpha > 0$, $\beta > 0$, the frequency parameter Ω for free plate is smaller than that for clamped plate and greater than that for simply supported plate irrespective of the value of other plate parameters. The frequency parameter Ω increases with increasing values of non-homogeneity parameter μ and taper parameters α and β, while it decreases with increasing value of density parameter η.

Figure 3.2(a) shows the effect of non-homogeneity parameter μ on the frequency parameter Ω for $\eta = 0.5$, $\alpha = 0.0, 0.3$ and $\beta = 0.0, \pm 0.3$ for all the three plates vibrating in fundamental mode. It is observed that frequency parameter increases with increasing value of non-homogeneity parameter μ for all the three cases. Also, the frequency parameter increases with increasing value of α or β or both for all the three plates. The increase is more pronounced in case of clamped plate as compared to simply supported and free plates. Figure 3.2(b) shows the plots for Ω versus μ for the second mode of vibration. It is observed that the rate of increase of Ω in all the three cases is higher than that in the fundamental mode. A similar behaviour can be seen from Figure 3.2(c) when the plate is vibrating in third mode.

Figure 3.3(a) depicts the variation of frequency parameter Ω with density parameter η for $\mu = 0.5$, $\alpha = 0.0, 0.3$ and $\beta = 0.0, \pm 0.3$ for all the three plates vibrating in fundamental mode. It is observed that frequency decreases with the increasing value of density parameter η. The rate of decrease with increasing value of η is more pronounced in case of free plate as compared to that of clamped or simply supported plate, whatever are the values of other plate parameters. A

similar inference is drawn when the plate is vibrating in second and third modes (Figures 3.3(b) and 3.3(c)).

Figures 3.4(a, b, c) show the effect of taper parameter α on frequency parameter Ω for μ = -0.5, 1.0, η = 0.5 and β = -0.3, 0.3 for plates vibrating in fundamental, second and third mode, respectively. It is observed that frequency parameter increases with increasing value of taper parameter α. The rate of increase of Ω is higher for clamped plate as compared to those for simply supported and free plates. Further, the frequency parameter can be increased / decreased by increasing / decreasing the value of β as well as of μ. The rate of increase becomes more pronounced with increase in number of modes.

Figures 3.5(a, b, c) show the plots of frequency parameter Ω versus taper parameter β for μ = -0.5, 1.0, η = 0.5 and taper parameter α = -0.3, 0.3 for plates vibrating in fundamental, second and third mode, respectively. It is found that frequency parameter increases with increasing value of taper parameter β except in case of free plate for α = -0.3. In this case, there appears a local minima in the vicinity of β = -0.3. This may be attributed to the increased mass of the plate towards the centre. However, for the second and third modes, the frequency parameter Ω is found to increase continuously with increasing value of β. The rate of increase of Ω with increasing value of β is higher for clamped plate as compared to those for simply supported and free plates.

Figures 3.6(a, b, c) show the plots of normalized transverse displacements for μ = -0.5, 1.0, η = 0.5, α = 0.0, β = 0.0; α = 0.5, β = 0.0 and α = 0.5, β = 0.5 for the first three modes of

vibration for clamped, simply supported and free plates, respectively. The radii of nodal circles decrease as the outer edge becomes thicker and thicker for all the three boundary conditions. The effect of non-homogeneity μ also decreases the radii of nodal circles.

Table 3.10 compares the results for homogeneous ($\mu = 0.0$, $\eta = 0.0$) circular plate of uniform thickness ($\alpha = 0.0$, $\beta = 0.0$) with exact solutions given by Leissa[1969] and approximate solutions obtained by Ansari[2000] using Ritz method and Azimi[1988] using receptence method. Table 3.11 gives a comparison of results for homogeneous circular plate of linearly varying thickness with those obtained by Lal[1979] using Frobenius method and with those obtained by Singh and Saxena[1995] and Gutierrez et al.[1996] using Rayleigh-Ritz method for clamped and simply-supported plate. A comparison of results for homogeneous circular plate of parabolically varying thickness with those obtained by Ansari[2000] using Ritz method, Lal[1979] using Frobenius method [31] and Gutierrez et al.[1996] using Rayleigh-Ritz method is presented in Table 3.12.

A comparative study for evaluation of frequency parameter Ω for a specified plate for the first three modes of vibration has been presented in Table 3.13 by taking equally spaced and three unequally spaced grid points i.e. zeros of shifted Chebyshev polynomials obtained from equations (2.3.9) and (2.3.10) and that of shifted Legendre polynomials. It is observed that zeros of Chebyshev polynomials provide comparatively faster rate of convergence.

Table 3.1

Values of frequency parameter Ω for clamped plate vibrating in fundamental mode

		η								
		-0.5			0			1		
		μ			μ			μ		
α	β	-0.5	0	1	-0.5	0	1	-0.5	0	1
-0.5	-0.1	5.1228	5.9564	8.0666	4.6587	5.4275	7.3794	3.8014	4.4474	6.0977
	0	5.7823	6.7376	9.1707	5.2676	6.1504	8.4058	4.3141	5.0589	6.9743
	0.1	6.4325	7.5089	10.2584	5.8686	6.8650	9.4184	4.8214	5.6651	7.8422
	0.5	8.9842	10.5316	14.4740	8.2329	9.6732	13.3572	6.8269	8.0606	11.2424
-0.1	-0.5	5.5010	6.3851	8.6287	4.9949	5.8090	7.8805	4.0629	4.7446	6.4891
	-0.1	8.1153	9.4993	13.0608	7.4095	8.6919	12.0029	6.0978	7.1856	10.0144
	0	8.7588	10.2651	14.1380	8.0052	9.4027	13.0084	6.6021	7.7906	10.8804
	0.1	9.3999	11.0270	15.2042	8.5992	10.1104	14.0048	7.1057	8.3942	11.7407
	0.5	11.9457	14.0407	19.3714	10.9619	12.9159	17.9094	9.1162	10.7964	15.1301
0	-0.5	6.2569	7.2797	9.8979	5.6889	6.6320	9.0529	4.6407	5.4328	7.4779
	-0.1	8.8576	10.3815	14.3059	8.0924	9.5055	13.1572	6.6688	7.8693	10.9949
	0	9.5005	11.1464	15.3791	8.6879	10.2158	14.1597	7.1733	8.4746	11.8597
	0.1	10.1415	11.9078	16.4420	9.2820	10.9235	15.1537	7.6774	9.0787	12.7191
	0.5	12.6893	14.9226	20.6014	11.6473	13.7310	19.0532	9.6912	11.4843	16.1078
0.1	-0.5	7.0065	8.1698	11.1639	6.3773	7.4513	10.2233	5.2144	6.1184	8.4663
	-0.1	9.6000	11.2643	15.5513	8.7756	10.3199	14.3121	7.2402	8.5540	11.9767
	0	10.2429	12.0288	16.6210	9.3712	11.0301	15.3121	7.7453	9.1598	12.8406
	0.1	10.8840	12.7901	17.6810	9.9657	11.7380	16.3041	8.2500	9.7646	13.6993
	0.5	13.4344	15.8064	21.8338	12.3341	14.5479	20.1993	10.2675	12.1739	17.0876
0.5	-0.5	9.9883	11.7224	16.2153	9.1180	10.7241	14.8998	7.5016	8.8627	12.4275
	-0.1	12.5786	14.8083	20.5411	11.5173	13.5905	18.9433	9.5348	11.3062	15.9195
	0	13.2232	15.5730	21.6005	12.1153	14.3021	19.9359	10.0431	11.9150	16.7809
	0.1	13.8667	16.3351	22.6519	12.7126	15.0119	20.9219	10.5514	12.5230	17.6381
	0.5	16.4301	19.3602	26.7851	15.0951	17.8331	24.8054	12.5840	14.9471	21.0272

Table 3.2

Values of frequency parameter Ω for clamped plate vibrating in second mode

		η								
		-0.5			0			1		
		μ			μ			μ		
α	β	-0.5	0	1	-0.5	0	1	-0.5	0	1
	-0.1	24.8132	28.4628	37.3364	21.8209	25.1062	33.1358	16.7503	19.3859	25.8941
-0.5	0	26.9194	30.8689	40.4535	23.7356	27.3002	35.9954	18.3158	21.1908	28.2756
	0.1	28.8978	33.1272	43.3760	25.5368	29.3624	38.6797	19.7932	22.8924	30.5168
	0.5	36.0125	41.2360	53.8464	32.0322	36.7861	48.3178	25.1528	29.0528	38.6035
	-0.5	26.8737	30.8487	40.5223	23.6421	27.2218	35.9814	18.1574	21.0308	28.1385
	-0.1	34.8338	39.9227	52.2236	30.8985	35.5198	46.7471	24.1237	27.8968	37.1580
-0.1	0	36.6128	41.9476	54.8301	32.5248	37.3763	49.1502	25.4691	29.4418	39.1805
	0.1	38.3363	43.9084	57.3526	34.1018	39.1755	51.4773	26.7760	30.9417	41.1420
	0.5	44.8131	51.2694	66.8073	40.0385	45.9409	60.2119	31.7153	36.6017	48.5267
	-0.5	29.5624	33.9258	44.5205	26.0750	30.0152	39.6350	20.1283	23.3085	31.1577
	-0.1	37.1829	42.6048	55.6962	33.0316	37.9627	49.9292	25.8648	29.9034	39.8041
0	0	38.9153	44.5753	58.2294	34.6170	39.7711	52.2669	27.1791	31.4115	41.7754
	0.1	40.5999	46.4904	60.6902	36.1597	41.5301	54.5389	28.4602	32.8806	43.6938
	0.5	46.9682	53.7238	69.9704	42.0016	48.1833	63.1185	33.3287	38.4559	50.9585
	-0.5	32.1408	36.8742	48.3462	28.4108	32.6950	43.1348	22.0253	25.4991	34.0563
	-0.1	39.4936	45.2416	59.1064	35.1312	40.3661	53.0564	27.5811	31.8804	42.4083
0.1	0	41.1865	47.1658	61.5773	36.6819	42.1337	55.3384	28.8692	33.3572	44.3360
	0.1	42.8374	49.0416	63.9846	38.1952	43.8579	57.5628	30.1281	34.7999	46.2173
	0.5	49.1098	56.1619	73.1101	43.9533	50.4120	66.0049	34.9341	40.3003	53.3756
	-0.5	41.7988	47.9016	62.6145	37.1793	42.7391	56.2127	29.1787	33.7466	44.9341
	-0.1	48.4622	55.4645	72.2966	43.2925	49.6977	65.1688	34.2723	39.5794	52.5251
0.5	0	50.0417	57.2557	74.5872	44.7438	51.3480	67.2899	35.4855	40.9669	54.3272
	0.1	51.5937	59.0152	76.8360	46.1705	52.9698	69.3732	36.6795	42.3318	56.0988
	0.5	57.5710	65.7867	85.4815	51.6715	59.2181	77.3894	41.2949	47.6025	62.9284

Table 3.3
Values of frequency parameter Ω for clamped plate vibrating in third mode

		η								
		-0.5			0			1		
		μ			μ			μ		
α	β	-0.5	0	1	-0.5	0	1	-0.5	0	1
-0.5	-0.1	58.2695	66.7093	86.9053	51.0286	58.5897	76.7771	38.8412	44.8512	59.4578
	0	62.6436	71.6301	93.0924	54.9910	63.0611	82.4331	42.0591	48.5050	64.1358
	0.1	66.7108	76.2016	98.8310	58.6805	67.2207	87.6853	45.0641	51.9135	68.4912
	0.5	81.0526	92.2944	118.9703	71.7238	81.8995	106.1599	55.7456	64.0058	83.8865
-0.1	-0.5	63.5808	72.7732	94.7416	55.7096	63.9543	83.7623	42.4408	49.0079	64.9530
	-0.1	79.7826	90.9644	117.5228	70.4288	80.5314	104.6456	54.4644	62.6327	82.3244
	0	83.3342	94.9452	122.4928	73.6637	84.1680	109.2117	57.1213	65.6374	86.1411
	0.1	86.7541	98.7763	127.2715	76.7811	87.6704	113.6050	59.6861	68.5361	89.8188
	0.5	99.4386	112.9704	144.9412	88.3630	100.6677	129.8735	69.2487	79.3295	103.4798
0	-0.5	69.3735	79.3069	102.9940	60.9308	69.8624	91.2716	46.6401	53.7902	71.1093
	-0.1	84.7370	96.5412	124.5395	74.9069	85.5877	111.0460	58.0881	66.7499	87.6003
	0	88.1704	100.3867	129.3342	78.0372	89.1041	115.4552	60.6645	69.6613	91.2929
	0.1	91.4901	104.1033	133.9642	81.0662	92.5050	119.7155	63.1613	72.4812	94.8657
	0.5	103.8901	117.9711	151.2091	92.3972	105.2133	135.6044	72.5320	83.0515	108.2282
0.1	-0.5	74.8842	85.5178	110.8268	65.9032	75.4844	98.4062	50.6483	58.3513	76.9708
	-0.1	89.5833	101.9935	131.3924	79.2904	90.5345	117.3012	61.6405	70.7839	92.7638
	0	92.9159	105.7237	136.0377	82.3316	93.9486	121.5765	64.1483	73.6158	96.3506
	0.1	96.1491	109.3412	140.5391	85.2840	97.2615	125.7217	66.5863	76.3675	99.8324
	0.5	108.2966	122.9194	157.4068	96.3925	109.7134	141.2736	75.7869	86.7399	112.9300
0.5	-0.5	95.2048	108.3867	139.5838	84.2756	96.2259	124.6483	65.5218	75.2493	98.6172
	-0.1	108.1749	122.8868	157.5961	96.1309	109.5183	141.2521	75.3295	86.3110	112.5917
	0	111.2158	126.2833	161.8083	98.9144	112.6360	145.1395	77.6391	88.9129	115.8719
	0.1	114.1921	129.6065	165.9274	101.6400	115.6878	148.9425	79.9030	91.4623	119.0837
	0.5	125.5564	142.2857	181.6230	112.0582	127.3436	163.4471	88.5764	101.2208	131.3580

Table 3.4
Values of frequency parameter Ω for simply supported plate vibrating in fundamental mode

		η								
		-0.5			0			1		
		μ			μ			μ		
α	β	-0.5	0	1	-0.5	0	1	-0.5	0	1
-0.5	-0.1	3.2709	3.7209	4.7788	2.9418	3.3516	4.3170	2.3426	2.6775	3.4697
	0	3.4712	3.9408	5.0531	3.1222	3.5498	4.5644	2.4868	2.8360	3.6678
	0.1	3.6597	4.1509	5.3248	3.2917	3.7388	4.8090	2.6216	2.9865	3.8628
	0.5	4.3703	4.9682	6.4540	3.9284	4.4716	5.8231	3.1239	3.5653	4.6660
-0.1	-0.5	3.6635	4.1625	5.3331	3.2928	3.7472	4.8150	2.6194	2.9906	3.8664
	-0.1	4.3996	4.9838	6.4014	3.9547	4.4859	5.7762	3.1459	3.5784	4.6317
	0	4.5717	5.1823	6.6773	4.1088	4.6637	6.0239	3.2674	3.7187	4.8275
	0.1	4.7431	5.3818	6.9600	4.2621	4.8423	6.2773	3.3880	3.8593	5.0276
	0.5	5.4356	6.2036	8.1625	4.8804	5.5767	7.3542	3.8724	4.4355	5.8760
0	-0.5	3.9578	4.4870	5.7368	3.5575	4.0392	5.1786	2.8303	3.2236	4.1571
	-0.1	4.6657	5.2871	6.8068	4.1930	4.7576	6.1401	3.3340	3.7932	4.9201
	0	4.8363	5.4854	7.0874	4.3455	4.9351	6.3917	3.4540	3.9330	5.1187
	0.1	5.0068	5.6856	7.3749	4.4980	5.1142	6.6493	3.5738	4.0738	5.3219
	0.5	5.7008	6.5136	8.5969	5.1172	5.8537	7.7432	4.0582	4.6532	6.1828
0.1	-0.5	4.2387	4.7996	6.1344	3.8098	4.3202	5.5364	3.0308	3.4471	4.4421
	-0.1	4.9294	5.5891	7.2151	4.4290	5.0281	6.5064	3.5200	4.0067	5.2101
	0	5.0992	5.7881	7.5007	4.5807	5.2061	6.7622	3.6392	4.1467	5.4118
	0.1	5.2696	5.9895	7.7933	4.7329	5.3860	7.0242	3.7585	4.2879	5.6182
	0.5	5.9661	6.8244	9.0342	5.3540	6.1313	8.1346	4.2439	4.8714	6.4914
0.5	-0.5	5.3019	6.0043	7.7289	4.7624	5.4002	6.9676	3.7838	4.3019	5.5774
	-0.1	5.9741	6.7982	8.8824	5.3626	6.1096	8.0002	4.2543	4.8585	6.3902
	0	6.1453	7.0035	9.1877	5.5152	6.2927	8.2732	4.3736	5.0018	6.6047
	0.1	6.3183	7.2118	9.4996	5.6694	6.4786	8.5521	4.4939	5.1471	6.8237
	0.5	7.0302	8.0765	10.8085	6.3033	7.2492	9.7218	4.9877	5.7486	7.7411

Table 3.5

Values of frequency parameter Ω for simply supported plate vibrating in second mode

		η								
		-0.5			0			1		
		μ			μ			μ		
α	β	-0.5	0	1	-0.5	0	1	-0.5	0	1
-0.5	-0.1	19.8930	22.7306	29.5751	17.3904	19.9308	26.0923	13.2080	15.2264	20.1746
	0	21.1646	24.1560	31.3380	18.5522	21.2386	27.7240	14.1696	16.3177	21.5609
	0.1	22.3520	25.4839	32.9734	19.6401	22.4601	29.2414	15.0751	17.3429	22.8573
	0.5	26.5982	30.2135	38.7592	23.5495	26.8323	34.6339	18.3632	21.0515	27.5109
-0.1	-0.5	21.9093	25.0752	32.7434	19.1638	21.9991	28.9061	14.5639	16.8165	22.3666
	-0.1	26.7425	30.4770	39.3748	23.5953	26.9741	35.0714	18.2565	20.9995	27.6512
	0	27.8135	31.6689	40.8279	24.5825	28.0774	36.4285	19.0880	21.9374	28.8264
	0.1	28.8502	32.8212	42.2302	25.5394	29.1458	37.7399	19.8968	22.8485	29.9655
	0.5	32.7467	37.1428	47.4713	29.1478	33.1649	42.6543	22.9674	26.2992	34.2600
0	-0.5	23.7449	27.1551	35.3799	20.8236	23.8870	31.3192	15.9085	18.3578	24.3697
	-0.1	28.3645	32.3083	41.6832	25.0686	28.6437	37.1925	19.4616	22.3759	29.4280
	0	29.4088	33.4691	43.0951	26.0326	29.7200	38.5132	20.2762	23.2937	30.5755
	0.1	30.4238	34.5961	44.4638	26.9708	30.7664	39.7950	21.0714	24.1887	31.6922
	0.5	34.2642	38.8520	49.6169	30.5313	34.7290	44.6324	24.1079	27.5988	35.9295
0.1	-0.5	25.5104	29.1533	37.9072	22.4222	25.7034	33.6357	17.2075	19.8452	26.2987
	-0.1	29.9658	34.1154	43.9589	26.5241	30.2927	39.2853	20.6541	23.7374	31.1841
	0	30.9879	35.2503	45.3366	27.4689	31.3465	40.5758	21.4547	24.6387	32.3086
	0.1	31.9845	36.3560	46.6770	28.3914	32.3743	41.8326	22.2385	25.5201	33.4065
	0.5	35.7763	40.5549	51.7534	31.9104	36.2880	46.6031	25.2459	28.8953	37.5942
0.5	-0.5	32.1785	36.6866	47.4039	28.4755	32.5691	42.3620	22.1519	25.4982	33.6053
	-0.1	36.2318	41.1810	52.8405	32.2285	36.7500	47.4659	25.3417	29.0862	38.0714
	0	37.1930	42.2449	54.1239	33.1209	37.7423	48.6734	26.1046	29.9426	39.1333
	0.1	38.1381	43.2902	55.3839	33.9993	38.7181	49.8597	26.8570	30.7865	40.1783
	0.5	41.7869	47.3208	60.2327	37.3972	42.4881	54.4319	29.7804	34.0603	44.2205

Table 3.6
Values of frequency parameter Ω for simply supported plate vibrating in third mode

		η								
		-0.5			0			1		
		μ			μ			μ		
α	β	-0.5	0	1	-0.5	0	1	-0.5	0	1
-0.5	-0.1	50.0849	57.2675	74.3921	43.7300	50.1461	65.5250	33.1102	38.1840	50.4760
	0	53.3298	60.8744	78.8013	46.6827	53.4404	69.5828	35.5298	40.9040	53.8779
	0.1	56.3533	64.2308	82.8930	49.4380	56.5106	73.3541	37.7949	43.4470	57.0496
	0.5	67.0534	76.0816	97.2716	59.2161	67.3808	86.6427	45.8806	52.5035	68.2906
-0.1	-0.5	55.1990	63.1298	82.0332	48.2169	55.3087	72.3047	36.5295	42.1477	55.7600
	-0.1	67.2196	76.4613	98.2501	59.1889	67.5232	87.2766	45.5782	52.2976	68.3937
	0	69.8657	79.3885	101.7923	61.6110	70.2127	90.5561	47.5876	54.5460	71.1776
	0.1	72.4167	82.2085	105.1999	63.9482	72.8061	93.7136	49.5302	56.7178	73.8627
	0.5	81.9027	92.6791	117.8154	72.6545	82.4521	105.4230	56.7944	64.8267	83.8563
0	-0.5	59.7779	68.2611	88.4161	52.3468	59.9533	78.1234	39.8575	45.9172	60.5504
	-0.1	71.1912	80.9034	103.7545	62.7797	71.5534	92.3053	48.4875	55.5859	72.5545
	0	73.7511	83.7329	107.1722	65.1256	74.1561	95.4730	50.4382	57.7665	75.2496
	0.1	76.2292	86.4701	110.4742	67.3983	76.6758	98.5357	52.3311	59.8811	77.8595
	0.5	85.5072	96.7042	122.7873	75.9212	86.1120	109.9742	59.4550	67.8279	87.6393
0.1	-0.5	64.1514	73.1585	94.4982	56.2954	64.3906	83.6731	43.0461	49.5259	65.1290
	-0.1	75.0832	85.2546	109.1409	66.3010	75.5037	97.2293	51.3445	58.8134	76.6342
	0	77.5698	88.0008	112.4526	68.5820	78.0323	100.3018	53.2451	60.9364	79.2537
	0.1	79.9848	90.6663	115.6632	70.7988	80.4883	103.2825	55.0952	63.0016	81.7986
	0.5	89.0786	100.6908	127.7083	79.1593	89.7388	114.4809	62.0949	70.8048	91.3888
0.5	-0.5	80.3894	91.3164	116.9831	70.9833	80.8729	104.2267	54.9546	62.9843	82.1519
	-0.1	90.0686	101.9913	129.8181	79.8778	90.7195	116.1565	62.3925	71.2816	92.3603
	0	92.3434	104.4969	132.8232	81.9714	93.0342	118.9535	64.1490	73.2383	94.7611
	0.1	94.5717	106.9502	135.7631	84.0234	95.3018	121.6913	65.8724	75.1572	97.1135
	0.5	103.0955	116.3255	146.9773	91.8815	103.9770	132.1453	72.4891	82.5168	106.1166

Table 3.6
Values of frequency parameter Ω for simply supported plate vibrating in third mode

		η								
		-0.5			0			1		
		μ			μ			μ		
α	β	-0.5	0	1	-0.5	0	1	-0.5	0	1
-0.5	-0.1	50.0849	57.2675	74.3921	43.7300	50.1461	65.5250	33.1102	38.1840	50.4760
	0	53.3298	60.8744	78.8013	46.6827	53.4404	69.5828	35.5298	40.9040	53.8779
	0.1	56.3533	64.2308	82.8930	49.4380	56.5106	73.3541	37.7949	43.4470	57.0496
	0.5	67.0534	76.0816	97.2716	59.2161	67.3808	86.6427	45.8806	52.5035	68.2906
-0.1	-0.5	55.1990	63.1298	82.0332	48.2169	55.3087	72.3047	36.5295	42.1477	55.7600
	-0.1	67.2196	76.4613	98.2501	59.1889	67.5232	87.2766	45.5782	52.2976	68.3937
	0	69.8657	79.3885	101.7923	61.6110	70.2127	90.5561	47.5876	54.5460	71.1776
	0.1	72.4167	82.2085	105.1999	63.9482	72.8061	93.7136	49.5302	56.7178	73.8627
	0.5	81.9027	92.6791	117.8154	72.6545	82.4521	105.4230	56.7944	64.8267	83.8563
0	-0.5	59.7779	68.2611	88.4161	52.3468	59.9533	78.1234	39.8575	45.9172	60.5504
	-0.1	71.1912	80.9034	103.7545	62.7797	71.5534	92.3053	48.4875	55.5859	72.5545
	0	73.7511	83.7329	107.1722	65.1256	74.1561	95.4730	50.4382	57.7665	75.2496
	0.1	76.2292	86.4701	110.4742	67.3983	76.6758	98.5357	52.3311	59.8811	77.8595
	0.5	85.5072	96.7042	122.7873	75.9212	86.1120	109.9742	59.4550	67.8279	87.6393
0.1	-0.5	64.1514	73.1585	94.4982	56.2954	64.3906	83.6731	43.0461	49.5259	65.1290
	-0.1	75.0832	85.2546	109.1409	66.3010	75.5037	97.2293	51.3445	58.8134	76.6342
	0	77.5698	88.0008	112.4526	68.5820	78.0323	100.3018	53.2451	60.9364	79.2537
	0.1	79.9848	90.6663	115.6632	70.7988	80.4883	103.2825	55.0952	63.0016	81.7986
	0.5	89.0786	100.6908	127.7083	79.1593	89.7388	114.4809	62.0949	70.8048	91.3888
0.5	-0.5	80.3894	91.3164	116.9831	70.9833	80.8729	104.2267	54.9546	62.9843	82.1519
	-0.1	90.0686	101.9913	129.8181	79.8778	90.7195	116.1565	62.3925	71.2816	92.3603
	0	92.3434	104.4969	132.8232	81.9714	93.0342	118.9535	64.1490	73.2383	94.7611
	0.1	94.5717	106.9502	135.7631	84.0234	95.3018	121.6913	65.8724	75.1572	97.1135
	0.5	103.0955	116.3255	146.9773	91.8815	103.9770	132.1453	72.4891	82.5168	106.1166

Table 3.7

Values of frequency parameter Ω for free plate vibrating in fundamental mode

		η								
		-0.5			0			1		
		μ			μ			μ		
α	β	-0.5	0	1	-0.5	0	1	-0.5	0	1
-0.5	-0.1	7.6304	8.4971	10.4984	6.5205	7.2797	9.0372	4.7589	5.3397	6.6913
	0	7.6512	8.5256	10.5669	6.5544	7.3210	9.1148	4.8080	5.3956	6.7773
	0.1	7.7159	8.6062	10.7077	6.6247	7.4059	9.2541	4.8812	5.4812	6.9070
	0.5	8.2370	9.2473	11.7358	7.1209	8.0104	10.2059	5.3145	6.0017	7.7045
-0.1	-0.5	8.6163	9.5738	11.7865	7.3760	8.2165	10.1628	5.3970	6.0422	7.5427
	-0.1	8.8145	9.8392	12.3049	7.5980	8.4992	10.6715	5.6378	6.3325	8.0127
	0	8.9357	9.9909	12.5548	7.7133	8.6418	10.9019	5.7385	6.4550	8.2046
	0.1	9.0754	10.1652	12.8378	7.8439	8.8034	11.1606	5.8494	6.5904	8.4169
	0.5	9.7741	11.0350	14.2256	8.4827	9.5950	12.4151	6.3716	7.2330	9.4247
0	-0.5	8.8437	9.8304	12.1336	7.5873	8.4539	10.4807	5.5762	6.2424	7.8065
	-0.1	9.1510	10.2270	12.8396	7.8992	8.8461	11.1490	5.8762	6.6069	8.3894
	0	9.2861	10.3962	13.1162	8.0258	9.0031	11.4019	5.9841	6.7388	8.5971
	0.1	9.4373	10.5851	13.4217	8.1657	9.1768	11.6798	6.1010	6.8823	8.8228
	0.5	10.1677	11.4956	14.8727	8.8302	10.0019	12.9879	6.6397	7.5474	9.8686
0.1	-0.5	9.1056	10.1284	12.5396	7.8267	8.7257	10.8485	5.7737	6.4656	8.1052
	-0.1	9.4973	10.6278	13.3955	8.2081	9.2034	11.6442	6.1191	6.8877	8.7782
	0	9.6442	10.8120	13.6951	8.3443	9.3728	11.9166	6.2332	7.0280	8.9996
	0.1	9.8055	11.0136	14.0203	8.4924	9.5570	12.2111	6.3554	7.1785	9.2370
	0.5	10.5643	11.9606	15.5277	9.1801	10.4125	13.5672	6.9093	7.8641	10.3170
0.5	-0.5	10.3456	11.5593	14.5223	8.9408	10.0093	12.6218	6.6624	7.4873	9.5095
	-0.1	10.9538	12.3267	15.7773	9.4999	10.7102	13.7572	7.1247	8.0607	10.4239
	0	11.1358	12.5551	16.1444	9.6649	10.9165	14.0871	7.2577	8.2260	10.6863
	0.1	11.3277	12.7956	16.5288	9.8382	11.1330	14.4320	7.3964	8.3985	10.9598
	0.5	12.1778	13.8588	18.2126	10.6010	12.0854	15.9387	8.0002	9.1506	12.1479

Table 3.8
Values of frequency parameter Ω for free plate vibrating in second mode

		η								
		-0.5			0			1		
		μ			μ			μ		
α	β	-0.5	0	1	-0.5	0	1	-0.5	0	1
-0.5	-0.1	27.7992	31.6458	40.7400	24.0956	27.5111	35.6358	18.0233	20.6983	27.1395
	0	28.9596	32.9017	42.1928	25.1820	28.6948	37.0232	18.9596	21.7315	28.3819
	0.1	30.0955	34.1363	43.6353	26.2416	29.8532	38.3921	19.8687	22.7369	29.5972
	0.5	34.3939	38.8314	49.1901	30.2381	34.2394	43.6260	23.2871	26.5254	34.2015
-0.1	-0.5	31.2974	35.6545	45.9448	27.1303	31.0036	40.2137	20.2879	23.3259	30.6470
	-0.1	35.6572	40.3737	51.4106	31.2257	35.4651	45.4439	23.8428	27.2474	35.3581
	0	36.7089	41.5197	52.7591	32.2078	36.5407	46.7212	24.6890	28.1839	36.4920
	0.1	37.7440	42.6490	54.0925	33.1735	37.5994	47.9816	25.5204	29.1045	37.6082
	0.5	41.7379	47.0157	59.2769	36.8960	41.6863	52.8669	28.7254	32.6549	41.9198
0	-0.5	33.2688	37.8352	48.5873	28.9275	33.0015	42.6582	21.7655	24.9847	32.7169
	-0.1	37.5521	42.4820	53.9981	32.9451	37.3860	47.8200	25.2476	28.8300	37.3480
	0	38.5853	43.6084	55.3258	33.9100	38.4432	49.0768	26.0797	29.7511	38.4637
	0.1	39.6035	44.7198	56.6394	34.8603	39.4853	50.3182	26.8990	30.6581	39.5635
	0.5	43.5453	49.0296	61.7578	38.5366	43.5210	55.1424	30.0688	34.1688	43.8251
0.1	-0.5	35.2184	39.9960	51.2167	30.7024	34.9779	45.0851	23.2220	26.6218	34.7652
	-0.1	39.4280	44.5697	56.5620	34.6476	39.2885	50.1744	26.6396	30.3984	39.3207
	0	40.4450	45.6790	57.8711	35.5977	40.3297	51.4131	27.4598	31.3063	40.4204
	0.1	41.4486	46.7747	59.1671	36.5348	41.3573	52.6378	28.2686	32.2017	41.5059
	0.5	45.3445	51.0343	64.2266	40.1704	45.3480	57.4078	31.4077	35.6776	45.7236
0.5	-0.5	42.8138	48.4339	61.5353	37.6092	42.6843	54.5883	28.8839	32.9955	42.7632
	-0.1	46.7930	52.7693	66.6391	41.3369	46.7652	59.4314	32.1185	36.5725	47.0865
	0	47.7623	53.8273	67.8907	42.2438	47.7593	60.6154	32.9048	37.4425	48.1396
	0.1	48.7224	54.8757	69.1326	43.1419	48.7440	61.7893	33.6835	38.3040	49.1827
	0.5	52.4791	58.9817	74.0079	46.6551	52.5980	66.3912	36.7318	41.6763	53.2667

Table 3.9
Values of frequency parameter Ω for free plate vibrating in third mode

		η								
		-0.5			0			1		
		μ			μ			μ		
α	β	-0.5	0	1	-0.5	0	1	-0.5	0	1
-0.5	-0.1	61.4057	70.1083	90.7096	53.4202	61.1637	79.5954	40.1876	46.2700	60.9077
	0	64.7300	73.7405	94.9876	56.4718	64.5148	83.5837	42.7252	49.0846	64.3257
	0.1	67.8744	77.1788	99.0476	59.3571	67.6851	87.3635	45.1252	51.7466	67.5604
	0.5	79.1947	89.5687	113.7115	69.7470	79.1060	101.0044	53.7854	61.3461	79.2270
-0.1	-0.5	68.4301	78.1543	101.1544	59.5351	68.1935	88.7900	44.7842	51.5928	67.9743
	-0.1	80.6464	91.4703	116.7667	70.7893	80.5231	103.3947	54.2121	62.0237	80.5794
	0	83.4122	94.4898	120.3227	73.3354	83.3153	106.7129	56.3458	64.3844	83.4353
	0.1	86.0926	97.4166	123.7717	75.8032	86.0217	109.9304	58.4149	66.6733	86.2044
	0.5	96.1418	108.3867	136.6994	85.0605	96.1719	121.9872	66.1887	75.2698	96.5943
0	-0.5	73.4365	83.6995	107.8806	64.0649	73.2322	94.9543	48.4547	55.7099	73.0974
	-0.1	85.1361	96.4536	122.8442	74.8469	85.0435	108.9485	57.4982	65.7119	85.1788
	0	87.8164	99.3786	126.2872	77.3162	87.7502	112.1628	59.5709	68.0039	87.9488
	0.1	90.4225	102.2231	129.6366	79.7173	90.3824	115.2889	61.5874	70.2335	90.6433
	0.5	100.2491	112.9441	142.2690	88.7770	100.3096	127.0717	69.2081	78.6556	100.8088
0.1	-0.5	78.2715	89.0574	114.3883	68.4391	78.0994	100.9146	51.9999	59.6869	78.0480
	-0.1	89.5401	101.3407	128.8027	78.8289	89.4787	114.3952	60.7265	69.3342	89.6932
	0	92.1468	104.1843	132.1480	81.2321	92.1118	117.5196	62.7468	71.5671	92.3892
	0.1	94.6882	106.9568	135.4108	83.5753	94.6791	120.5660	64.7178	73.7451	95.0185
	0.5	104.3165	117.4568	147.7847	92.4590	104.4080	132.1079	72.2024	82.0119	104.9848
0.5	-0.5	96.4352	109.1903	138.8661	84.8795	96.3929	123.3272	65.3467	74.6542	96.6718
	-0.1	106.5158	120.1680	151.7385	94.1945	106.5817	135.3761	73.2131	83.3344	107.1164
	0	108.9058	122.7713	154.7962	96.4036	108.9980	138.2355	75.0805	85.3944	109.5942
	0.1	111.2525	125.3273	157.8007	98.5729	111.3705	141.0440	76.9154	87.4180	112.0275
	0.5	120.2662	135.1472	169.3792	106.9092	120.4855	151.8515	83.9767	95.2014	121.3808

Table 3.10

Comparison of frequency parameter Ω for homogeneous ($\mu = 0.0$, $\eta = 0.0$) circular plate of uniform thickness ($\alpha = 0.0$, $\beta = 0.0$)

Mode	$\nu = 0.3$				$\nu = 0.33$	
	Clamped plate		S-S plate		Free plate	
I	10.2158 10.2158°	10.2158* 10.216◊	4.9351 4.9352°	4.977* 4.935◊	9.0689	9.084*
II	39.7711 39.7711°	39.771* 39.771◊	29.7200 29.7200°	29.76* 29.720◊	38.507	38.55*
III	89.1041 89.1041°	89.104* 89.103◊	74.1561 74.1961°	74.20* 74.156◊	87.8127	87.80*

* Values taken from Lal[1979].
° Values taken from Ansari[2000].
◊ Values taken from Gutierrez et al.[1996].

Table 3.11

Comparison of frequency parameter Ω for homogeneous ($\mu = 0.0$, $\eta = 0.0$) circular plate of linear thickness variation ($\beta = 0.0$)

α	Clamped			S-S		
	I	II	III	I	II	III
-0.5	6.1504 6.1522* 6.1504°	27.3002 27.3006* 27.300°	63.0611 63.0605* 63.062°	3.5498 3.5507* 3.5498°	21.2386 21.2419* 21.239°	53.4404 53.4095* 53.441°
-0.3	7.7783 7.7769* 7.7783° 7.778$^{\diamond}$	32.4610 32.4586* 32.461° 32.463$^{\diamond}$	73.9467 73.9586* 73.947°	4.1158 4.1154* 4.1158° 4.116$^{\diamond}$	24.7265 24.7268* 24.727° 24.728$^{\diamond}$	62.0704 62.0732* 62.071°
-0.1	9.4027 9.4016* 9.4027° 9.402$^{\diamond}$	37.3763 37.3742* 37.376° 37.376$^{\diamond}$	84.1680 84.1188* 84.168°	4.6637 4.6627* 4.6637° 4.664$^{\diamond}$	28.0774 28.0765* 28.077° 28.078$^{\diamond}$	70.2127 70.2104* 70.213°
0.1	11.0301 11.0297* 11.030° 11.03$^{\diamond}$	42.1337 42.1408* 42.134° 42.133$^{\diamond}$	93.9486 93.9014* 93.949°	5.2061 5.2065* 5.2061° 5.206$^{\diamond}$	31.3465 31.3467* 31.346° 31.346$^{\diamond}$	78.0323 78.0254* 78.032°
0.3	12.6631 12.6648* 12.663° 12.663$^{\diamond}$	46.7813 46.7965* 46.782° 46.784$^{\diamond}$	103.4123 103.8434* 103.41°	5.7483 5.7469* 5.7483° 5.748$^{\diamond}$	34.5625 34.5613* 34.563° 34.564$^{\diamond}$	85.6205 85.5148* 85.623°
0.5	14.3021 14.3033* 14.302°	51.3480 51.3588* 51.349°	112.6360 112.4586* 112.64°	6.2927 6.2908* 6.2928°	37.7423 37.7414* 37.743°	93.0342 92.7375* 93.042°

* Values taken from Lal[1979].
$^{\circ}$ Values taken from Singh and Saxena[1995].
$^{\diamond}$ Values taken from Gutierrez et al.[1996].

Table 3.12

Comparison of frequency parameter Ω for homogeneous ($\mu = 0.0, \eta = 0.0$) circular plate of parabolic thickness variation ($\alpha = 0.0$)

β	Clamped			S-S		
	I	II	III	I	II	III
-0.5	6.6320 6.6303*	30.0152 30.0130*	69.8624 69.8709*	4.0392 4.0391*	23.887 23.8884*	59.9533 59.9567*
-0.3	8.0759 8.0748* 8.0759° 8.076$^{\Diamond}$	34.161 34.1768* 34.1610° 34.161$^{\Diamond}$	78.1241 78.1086*	4.4034 4.4029* 4.4034° 4.403$^{\Diamond}$	26.3765 26.3757* 26.3765° 26.376$^{\Diamond}$	66.0394 66.0258*
-0.1	9.5055 9.5055* 9.5055° 9.505$^{\Diamond}$	37.9627 37.9631* 37.9627° 37.963$^{\Diamond}$	85.5877 85.5598*	4.7576 4.7562* 4.7576° 4.758$^{\Diamond}$	28.6437 28.6447* 28.6437° 28.644$^{\Diamond}$	71.5534 71.5579*
0.1	10.9235 10.9223* 10.9235° 10.924$^{\Diamond}$	41.5301 41.5380* 41.5301° 41.529$^{\Diamond}$	92.505 92.5123*	5.1142 5.1130* 5.1142° 5.114$^{\Diamond}$	30.7664 30.7682* 30.7664° 30.768$^{\Diamond}$	76.6758 76.6675*
0.3	12.3317 12.3287* 12.3317° 12.332$^{\Diamond}$	44.9242 44.9329* 44.9242° 44.921$^{\Diamond}$	99.0172 99.2534*	5.4787 5.4802* 5.4787° 5.479$^{\Diamond}$	32.7863 32.7877* 32.7863° 32.786$^{\Diamond}$	81.5074 81.5172*
0.5	13.731 13.7317*	48.1833 48.1822*	105.2133 —	5.8537 5.8509*	34.729 34.7138*	86.112 —

* Values taken from Lal[1979].
° Values taken from Ansari[2000].
$^{\Diamond}$ Values taken from Gutierrez et al.[1996].

Table 3.13

Number of grid points for convergence of frequency parameter Ω by using zeros of Chebyshev polynomial, Legendre polynomial and equidistant collocation points for clamped plate for $\eta = 0.5$, $\mu = -0.5$

	$\alpha = -0.5,\ \beta = 0.5$			$\alpha = 0.0,\ \beta = 0.5$			$\alpha = 0.5,\ \beta = -0.5$			$\alpha = 0.5,\ \beta = 0.5$		
	I	II	III	I	II	III	I	II	III	I	II	III
Ω / Grid Points	7.5131	28.4203	63.3102	10.6470	37.4630	81.9673	8.2884	32.9813	74.4059	13.8119	46.2540	99.7549
Zeros of Chebyshev polynomial	13	15	17	11	13	16	11	12	14	11	12	15
Zeros of Legendre polynomial	13	15	17	11	14	17	11	14	16	11	13	16
Equidistant	17	22	24	16	20	23	16	19	22	13	18	21
Liew et al.[1997]	13	14	18	12	15	18	12	15	16	12	15	16

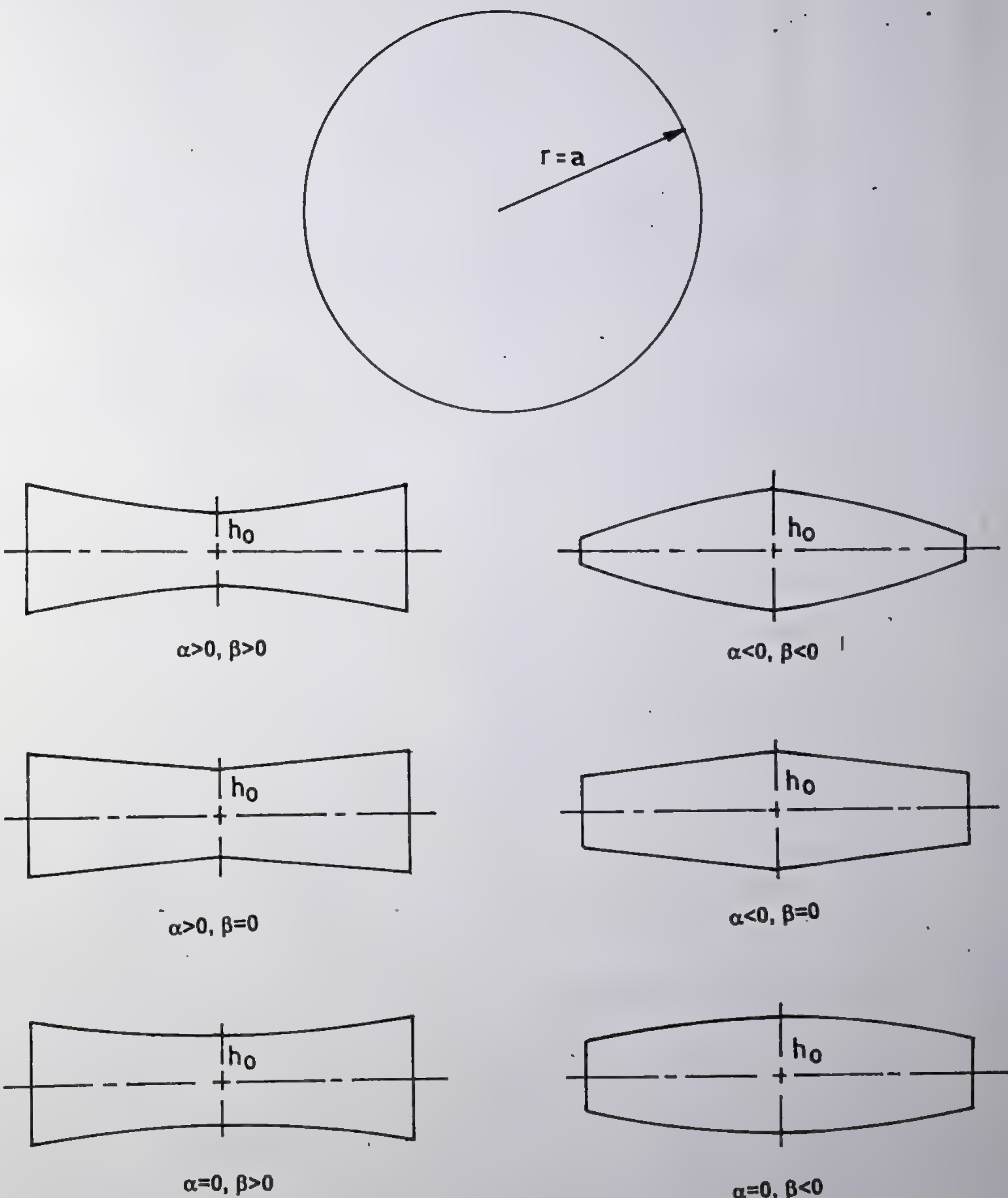

Fig. 3 : Geometry and cross-section of the tapered circular plate for quadratic thickness variation i.e. $\bar{h} = h_0\left(1 + \alpha x + \beta x^2\right)$ where $x = \dfrac{r}{a}$

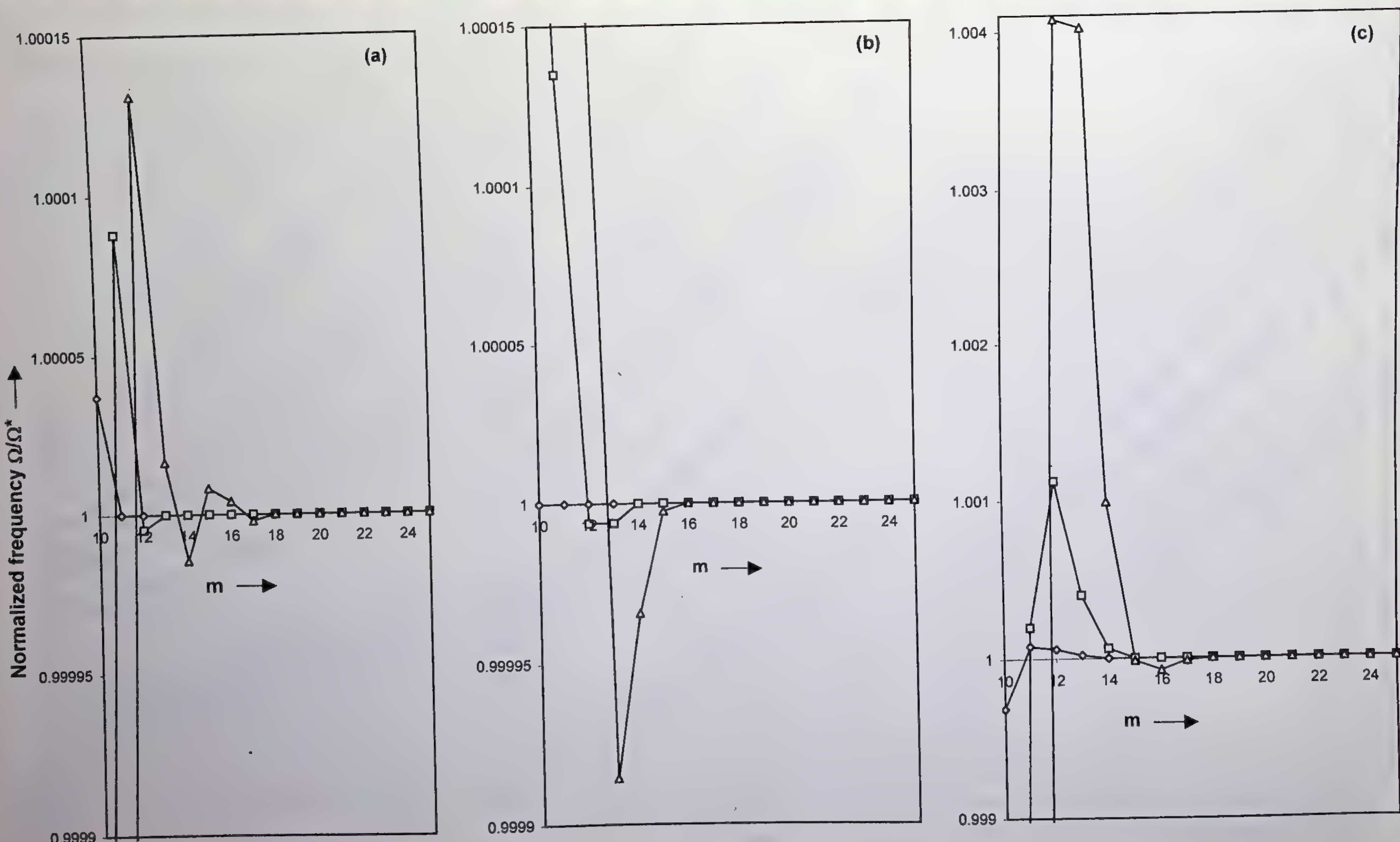

Fig. 3.1 : Convergence of the normalized frequency parameter Ω/Ω^* for the first three modes of vibration with grid refinement for $\eta = 1.0$, $\mu = -0.5$, $\alpha = -0.4$, $\beta = 0.1$ for (a) Clamped (b) Simply supported and (c) Free plate.
—◇—, fundamental mode; —□—, second mode; —Δ—, third mode. Ω^*- the DQ results using 25 grid points.

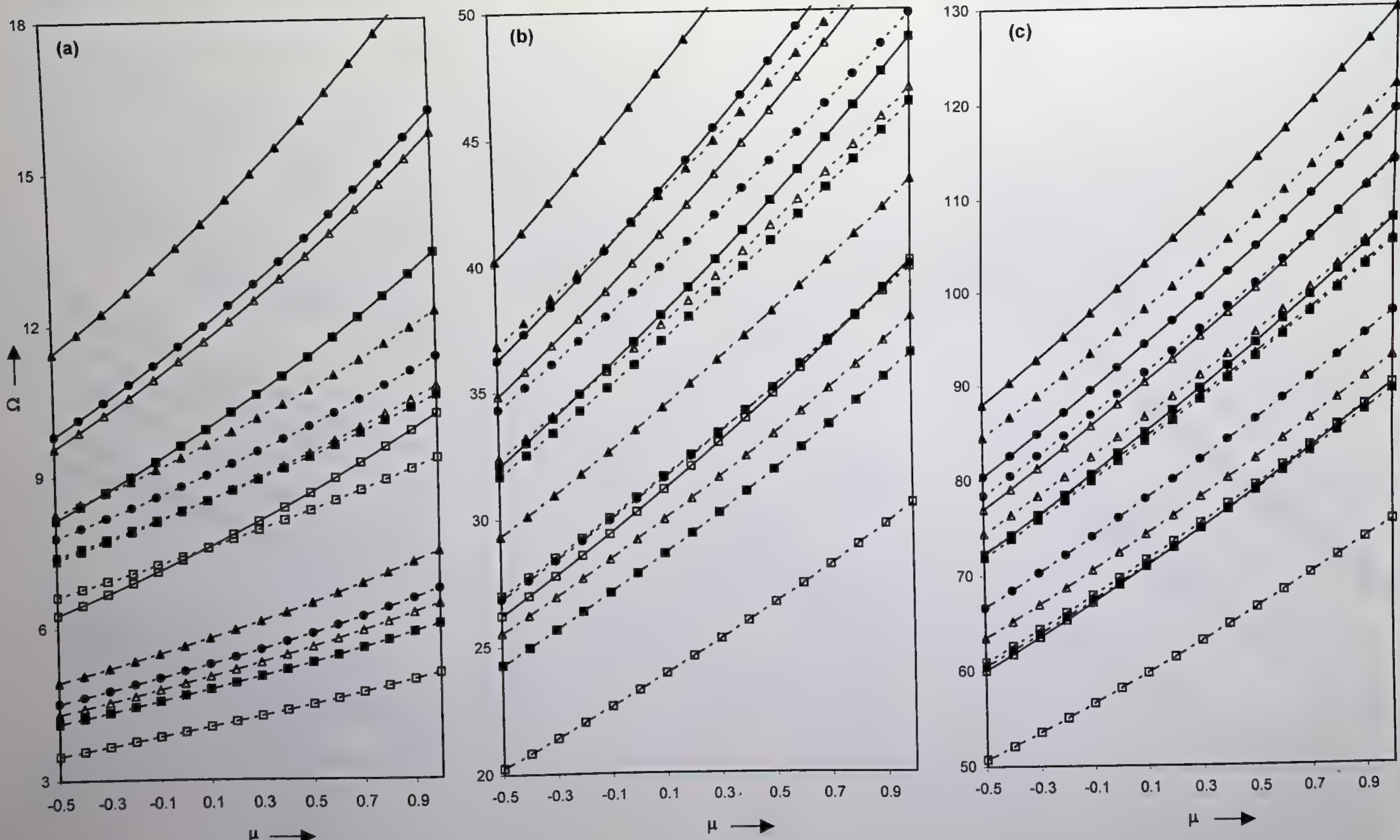

Fig. 3.2 : Frequency parameter for clamped, simply supported and free plates vibrating in (a) fundamental (b) second and (c) third mode for $\eta = 0.5$. ———, clamped ; - - - - - - -, simply supported ; ------------, free.
□, $\alpha = 0, \beta = -0.3$; Δ, $\alpha = 0, \beta = 0.3$; ■, $\alpha = 0.3, \beta = -0.3$; ●, $\alpha = 0.3, \beta = 0$; ▲, $\alpha = 0.3, \beta = 0.3$.

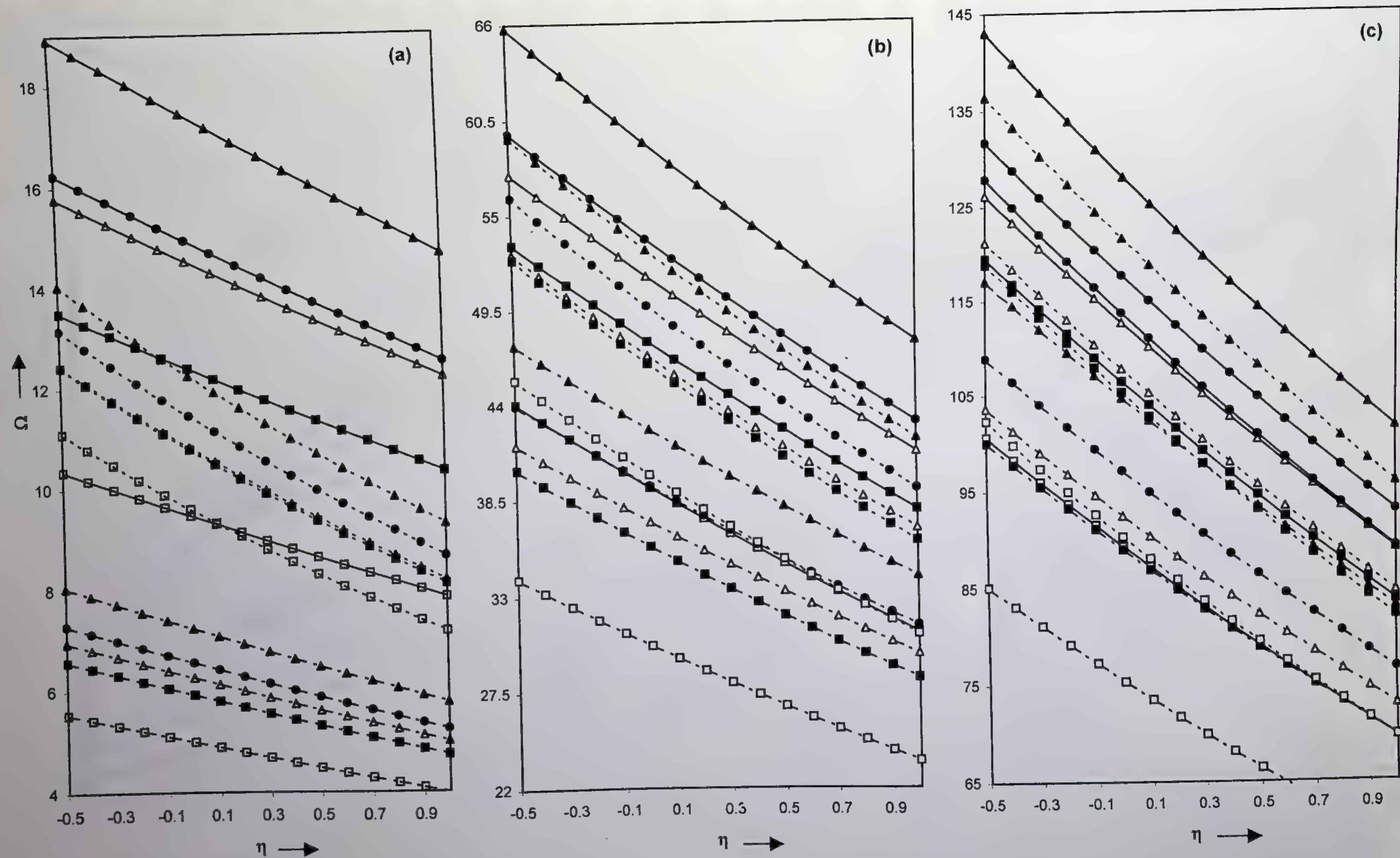

Fig. 3.3 : Frequency parameter for clamped, simply supported and free plates vibrating in (a) fundamental (b) second and (c) third mode for $\mu = 0.5$. ———, clamped ; – – – – – –, simply supported ; ------------, free.
□, $\alpha = 0$, $\beta = -0.3$; Δ, $\alpha = 0$, $\beta = 0.3$; ■, $\alpha = 0.3$, $\beta = -0.3$; ●, $\alpha = 0.3$, $\beta = 0$; ▲, $\alpha = 0.3$, $\beta = 0.3$.

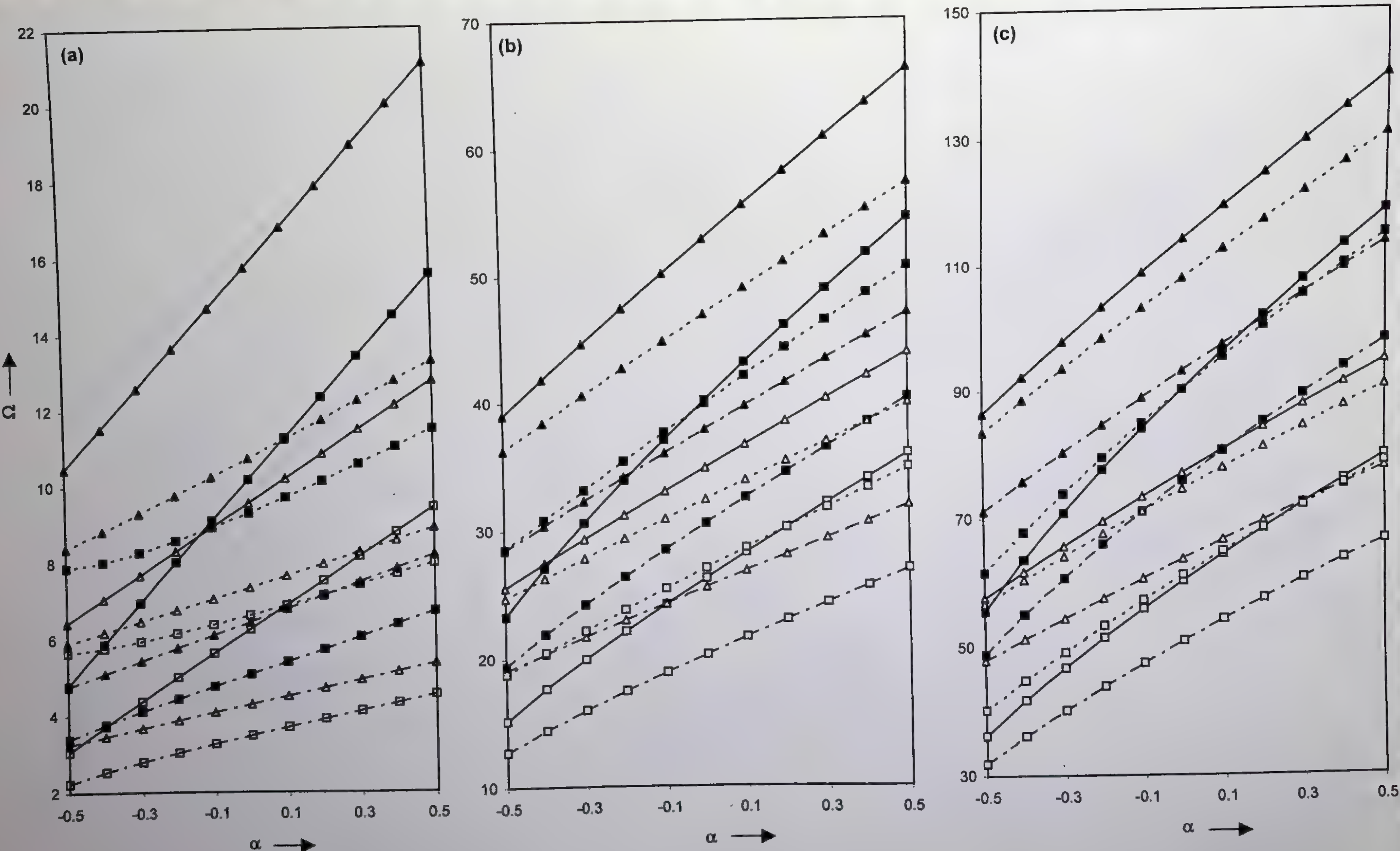

Fig. 3.4 : Frequency parameter for clamped, simply supported and free plates vibrating in (a) fundamental (b) second and (c) third mode for $\eta = 0.5$. ———, clamped ; – – – – – – –, simply supported ; ------------, free.
□, $\mu = -0.5$, $\beta = -0.3$; Δ, $\mu = -0.5$, $\beta = 0.3$; ■, $\mu = 1.0$, $\beta = -0.3$; ▲, $\mu = 1.0$, $\beta = 0.3$.

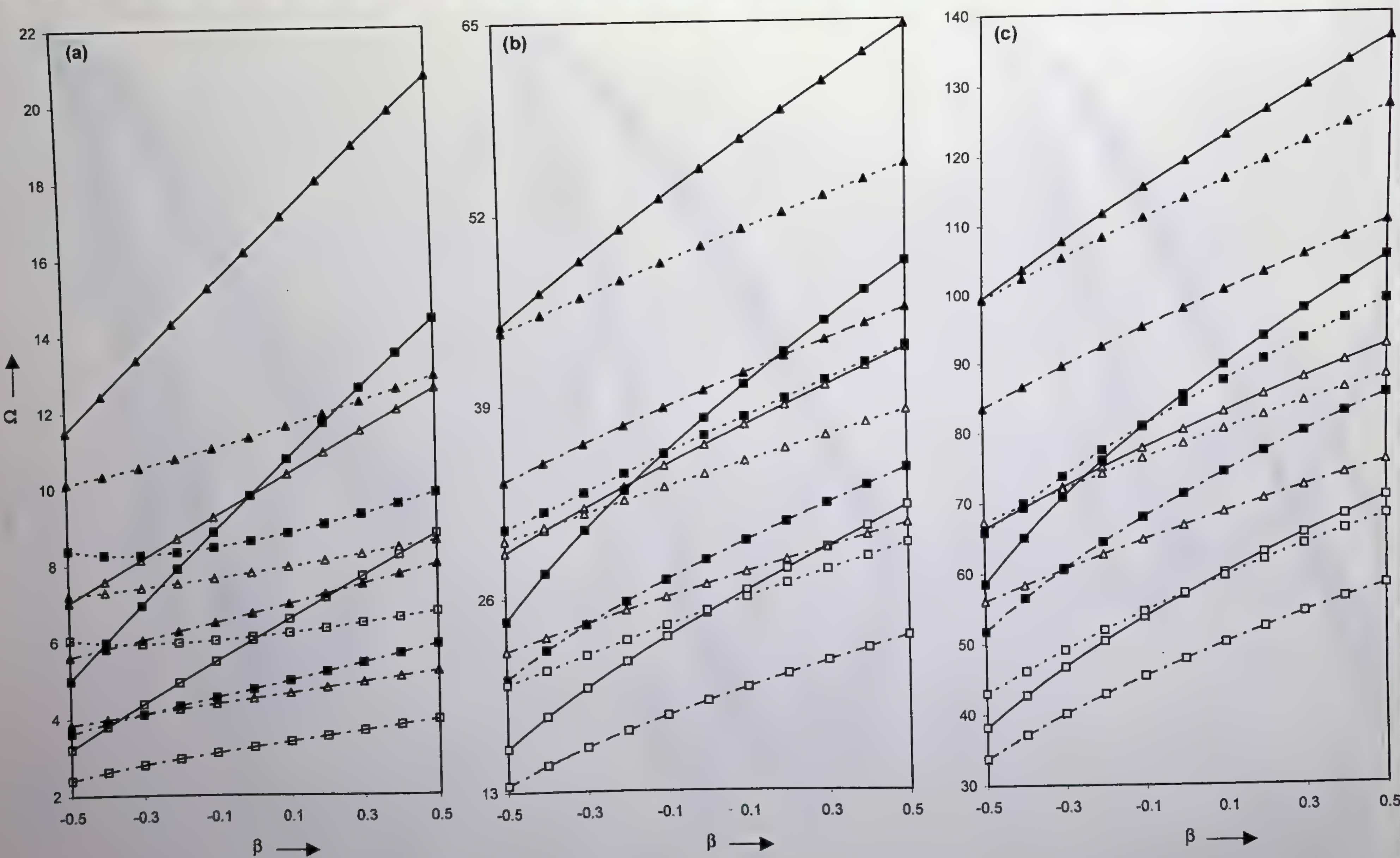

Fig. 3.5 : Frequency parameter for clamped, simply supported and free plates vibrating in (a) fundamental (b) second and (c) third mode for $\eta = 0.5$. ———, clamped ; – – – – – – –, simply supported ; ------------, free.
□, $\mu = -0.5$, $\alpha = -0.3$; Δ, $\mu = -0.5$, $\alpha = 0.3$; ■, $\mu = 1.0$, $\alpha = -0.3$; ▲, $\mu = 1.0$, $\alpha = 0.3$.

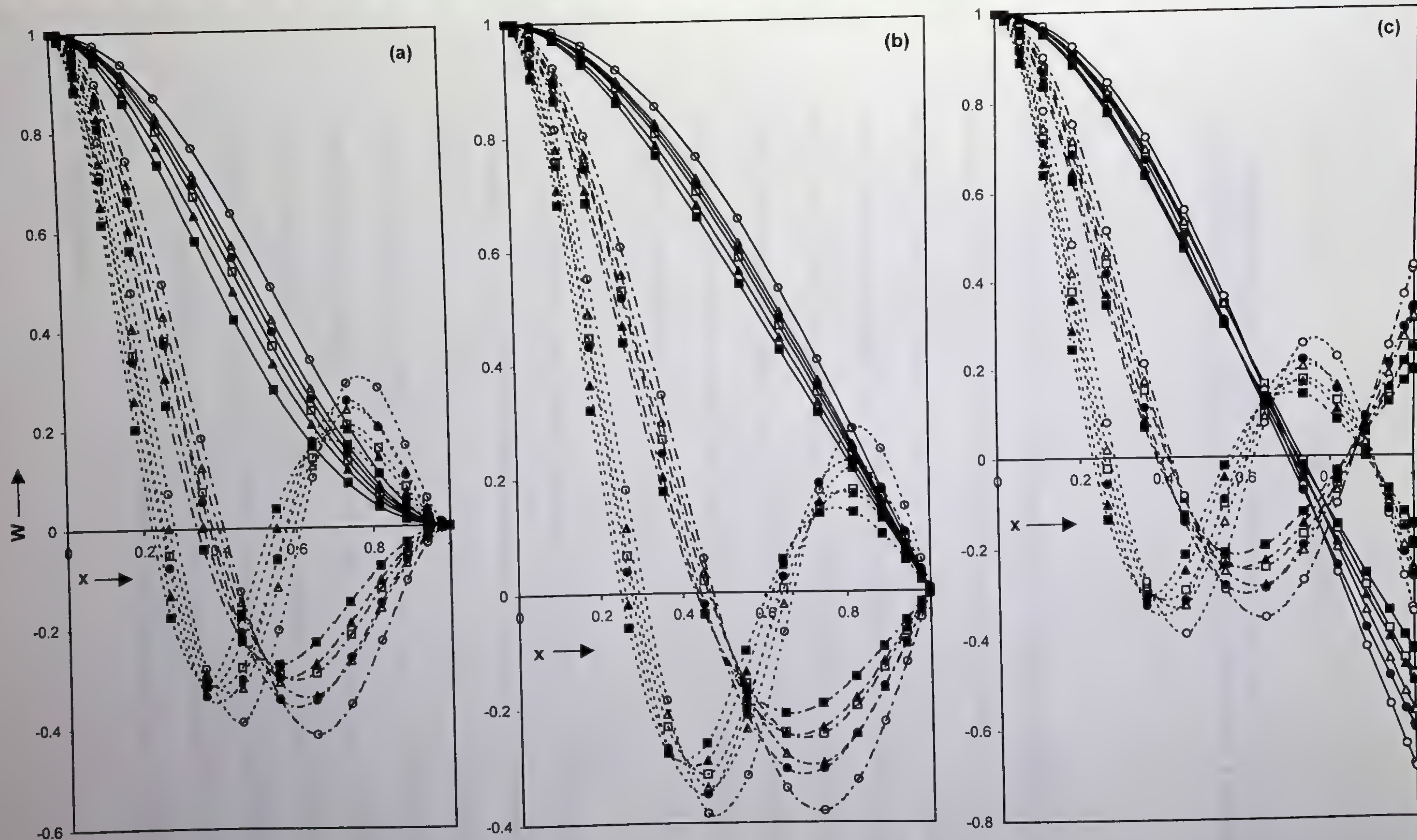

Fig. 3.6: Normalized displacements for the first three modes of vibration for (a) clamped (b) simply supported (c) free plate for $\eta = 0.5$.

———, fundamental mode; - - - - - - - -, second mode; ------------, third mode.

□, $\alpha = 0.5$, $\beta = 0.5$; Δ, $\alpha = 0.5$, $\beta = 0$; ○, $\alpha = 0$, $\beta = 0$. □, Δ, ○, $\mu = -0.5$; ■, ▲, ●, $\mu = 1.0$.

CHAPTER IV

AXISYMMETRIC VIBRATIONS OF NON-HOMOGENEOUS MINDLIN'S ANNULAR PLATES OF VARIABLE THICKNESS

1. INTRODUCTION

Plates of variable thickness are widely preferred in structural engineering particularly in aerospace industry and ocean engineering system due to the desirability of reduction in weight and size. A number of studies dealing with vibration of isotropic homogeneous plates of variable thickness is available in the literature. However, a very little work (Tomar[1982a, 1982b, 1983, 1984] is available analyzing vibration of isotropic non-homogeneous plates of variable thickness. As the plates used in various applications may have appreciable thickness, so the effect of shear deformation and rotatory inertia cannot be neglected (Mindlin [1951], Deresiewicz and Mindlin [1955]).

This chapter deals with axisymmetric vibrations of non-homogeneous Mindlin's annular plate of quadratically varying thickness. The governing differential equation with regard to vibration of such plates has been derived by Hamilton's energy principle. The inclusion of transverse shear and rotatory inertia along with non-homogeneity leads to a set of coupled differential equations with variable coefficients, whose analytical solution is not feasible. Therefore, an approximate solution has been obtained by employing differential quadrature method. Frequencies have been computed for different values of various plate parameters for three different sets of boundary conditions. Mode shapes for the first three modes of vibration have been obtained for specified plates. A comparison of results obtained by classical plate theory has been presented.

2. BASIC PLATE EQUATIONS

Consider an isotropic homogeneous annular plate of thickness $h(r)$ with inner and outer radii b and a, respectively, referred to a system of cylindrical coordinates $(r,\ \theta,\ z)$, where the axis of the plate is taken as the line $r = 0$ and its middle surface as the plane $z = 0$.

Strain- Displacement Relations

Let $(u,\ v,\ w)$ be the displacement components at a point $(r,\ \theta,\ z)$ in r, θ and z-directions respectively. We assume that u and v are proportional to z and w is independent of z. For axisymmetric vibrations, the displacement will also be axisymmetric and hence $\dfrac{\partial(\ )}{\partial\theta} = 0$.

Therefore, the kinematic relations between the displacement components are given by

$$
\begin{aligned}
u &= z\ \psi_r(r,t),\\
v &= 0\ ,\\
w &= w(r,t)\ ,
\end{aligned}
\tag{4.2.1}
$$

where ψ_r is the angle of rotation of the plate element in r-z plane. The strain components in terms of displacements (Love [1944], p.56) become

$$
\begin{aligned}
\varepsilon_r &= z\frac{\partial \psi_r}{\partial r}\ ,\\
\varepsilon_\theta &= \frac{z}{r}\psi_r\ ,\\
\varepsilon_{rz} &= \psi_r + \frac{\partial w}{\partial r}\ ,\\
\varepsilon_{r\theta} &= \varepsilon_{\theta z} = 0\ .
\end{aligned}
\tag{4.2.2}
$$

Stress-Strain Relations

The stress-strain relations for the isotropic material are as follows

$$\sigma_r = \frac{E}{1-\upsilon^2}\left[\varepsilon_r + \upsilon\varepsilon_\theta\right], \tag{4.2.3}$$

$$\sigma_\theta = \frac{E}{1-\upsilon^2}\left[\varepsilon_\theta + \upsilon\varepsilon_r\right],$$

$$\sigma_{rz} = G\,\varepsilon_{rz}\ ,$$

where, E is the Young's modulus, υ the Poisson's ratio and G is the shear modulus.

Moment and Shear Resultants

If Q_r, M_r and M_θ denote the transverse shear resultant and moment resultants, all per unit length, then

$$Q_r = \int_{-h/2}^{h/2} \sigma_{rz}\, dz\ , \tag{4.2.4}$$

$$\left(M_r, M_\theta\right) = \int_{-h/2}^{h/2} (\sigma_r, \sigma_\theta)\, z\, dz\ ,$$

where $z = \pm h/2$ are the lower and upper faces of the plate.

Energy Variations

The strain energy density is given by

$$dW = \frac{1}{2}\left[\sigma_r\varepsilon_r + \sigma_\theta\varepsilon_\theta + \sigma_{r\theta}\varepsilon_{r\theta} + \sigma_{rz}\varepsilon_{rz} + \sigma_{\theta z}\varepsilon_{\theta z}\right] dV\ , \tag{4.2.5}$$

where dV denotes elementary volume.

For axisymmetric case, the total strain energy of the plate is obtained by integrating the relation (4.2.5) over the total volume of the plate and is given by

$$W = \frac{1}{2}\int_{b}^{a}\int_{0}^{2\pi}\int_{-h/2}^{h/2}\left[\sigma_r\varepsilon_r + \sigma_\theta\varepsilon_\theta + \sigma_{rz}\varepsilon_{rz}\right] r\,dz\,d\theta\,dr \ . \tag{4.2.6}$$

Substituting for strains from relation (4.2.2) we get,

$$W = \frac{1}{2}\int_{b}^{a}\int_{0}^{2\pi}\int_{-h/2}^{h/2}\left[\sigma_r\left(z\frac{\partial\psi_r}{\partial r}\right) + \sigma_\theta\frac{z}{r}\psi_r + \sigma_{rz}\left(\psi_r + \frac{\partial w}{\partial r}\right)\right] r\,dz\,d\theta\,dr \ . \tag{4.2.7}$$

Integrating with respect to z and substituting the values from (4.2.4), we get

$$W = \int_{b}^{a}\int_{0}^{2\pi}\left[M_r\frac{\partial\psi_r}{\partial r} + \frac{M_\theta}{r}\psi_r + Q_r\left(\psi_r + \frac{\partial w}{\partial r}\right)\right] r\,d\theta\,dr \ . \tag{4.2.8}$$

The expression for kinetic energy is given by

$$dT = \frac{\rho}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] dV \ , \tag{4.2.9}$$

where, ρ is the volume density of the plate material.

The total kinetic energy of the plate is obtained by integrating the relation (4.2.9) over the total volume of the plate i.e.

$$T = \frac{1}{2}\int_{b}^{a}\int_{0}^{2\pi}\int_{-h/2}^{h/2}\rho\left[\left(z\frac{\partial\psi_r}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] r\,dz\,d\theta\,dr \ . \tag{4.2.10}$$

Integrating with respect to z, we get

$$T = \frac{1}{2}\int_b^a \int_0^{2\pi} \rho \left[\frac{h^3}{12}\left(\frac{\partial \psi_r}{\partial t}\right)^2 + h\left(\frac{\partial w}{\partial t}\right)^2 \right] r\, d\theta\, dr \,. \tag{4.2.11}$$

Now the variations of the expressions (4.2.8) and (4.2.11) are

$$\delta W = \int_b^a \int_0^{2\pi} \left[M_r \frac{\partial\,(\delta\psi_r)}{\partial r} + M_\theta \frac{\delta\psi_r}{r} + Q_r\left((\delta\psi_r) + \frac{\partial(\delta w)}{\partial r} \right) \right] r\, d\theta dr \;, \tag{4.2.12}$$

$$\delta T = \int_b^a \int_0^{2\pi} \rho \left[\frac{h^3}{12}\left(\frac{\partial \psi_r}{\partial t}\frac{\partial(\delta\psi_r)}{\partial t} \right) + h\left(\frac{\partial w}{\partial t}\frac{\partial(\delta w)}{\partial t} \right) \right] r\, d\theta\, dr \;. \tag{4.2.13}$$

Equation of Motion

According to Hamilton's energy principle,

$$\delta \int_{t_1}^{t_2} L dt = 0 \quad , \tag{4.2.14}$$

where t_1 and t_2 are the initial and final values of time and the kinetic potential $L = T - W$.

Taking the variational operator δ inside the integral (4.2.14) and using the equations (4.2.12) and (4.2.13), we get

$$\int_b^a \int_0^{2\pi} \int_{t_1}^{t_2} \left[\begin{array}{c} M_r \dfrac{\partial(\delta\psi_r)}{\partial r} + \dfrac{M_\theta}{r}\delta\psi_r + Q_r\left(\delta\psi_r + \dfrac{\partial(\delta w)}{\partial r}\right) \\ -\rho\left\{ \begin{array}{l} \left(\dfrac{h^3}{12}\dfrac{\partial\psi_r}{\partial t}\dfrac{\partial(\delta\psi_r)}{\partial t}\right) \\ +\left(h\dfrac{\partial w}{\partial t}\dfrac{\partial(\delta w)}{\partial t}\right) \end{array} \right\} \end{array} \right] r\, dt\, d\theta\, dr = 0, \tag{4.2.15}$$

or

$$\int_b^a \int_0^{2\pi} \int_{t_1}^{t_2} \left[\begin{array}{c} rM_r \dfrac{\partial(\delta\psi_r)}{\partial r} + M_\theta \delta\psi_r + rQ_r\left(\delta\psi_r + \dfrac{\partial(\delta w)}{\partial r}\right) \\ -\rho r \left(\begin{array}{c} \dfrac{h^3}{12}\dfrac{\partial\psi_r}{\partial t}\dfrac{\partial(\delta\psi_r)}{\partial t} \\ +h\dfrac{\partial w}{\partial t}\dfrac{\partial(\delta w)}{\partial t} \end{array} \right) \end{array} \right] dt\, d\theta\, dr = 0. \tag{4.2.16}$$

Integrating equation (4.2.16) by parts, the integrated part gives the boundary conditions and the remaining triple integrals are

$$\int_b^a \int_0^{2\pi} \int_{t_1}^{t_2} \left[\begin{array}{c} \left(M_r + r\dfrac{\partial M_r}{\partial r} - M_\theta - rQ_r - \dfrac{\rho h^3}{12} r \dfrac{\partial^2 \psi_r}{\partial t^2} \right)\delta\psi_r \\ +\left(Q_r + r\dfrac{\partial Q_r}{\partial r} - \rho r h \dfrac{\partial^2 w}{\partial t^2} \right)\delta w \end{array} \right] dt\, d\theta\, dr = 0. \tag{4.2.17}$$

Expression (4.2.17) will be satisfied only when the coefficients of $\delta\psi_r$ and δw are zero separately and hence, we get

$$\frac{\partial M_r}{\partial r} + \frac{M_r - M_\theta}{r} - Q_r - \frac{\rho h^3}{12}\frac{\partial^2 \psi_r}{\partial t^2} = 0 \ , \tag{4.2.18}$$

$$\frac{1}{r}Q_r + \frac{\partial Q_r}{\partial r} - \rho h \frac{\partial^2 w}{\partial t^2} = 0 \ , \tag{4.2.19}$$

which are the required plate equations of motion.

For elastically non-homogeneous plates of variable thickness $h(r)$, the moment and shear resultants (Deresiewicz and Mindlin [1955]) are given by

$$M_r = D\left(\frac{\partial \psi_r}{\partial r} + \frac{\upsilon}{r}\psi_r\right),$$
$$M_\theta = D\left(\frac{\psi_r}{r} + \upsilon\frac{\partial \psi_r}{\partial r}\right), \tag{4.2.20}$$
$$Q_r = \kappa G h\left(\psi_r + \frac{\partial w}{\partial r}\right),$$

where $D = \dfrac{E(r)h^3(r)}{12(1-\upsilon^2)}$ is the flexural rigidity , $\kappa = \dfrac{\pi^2}{12}$ is an averaging shear coefficient and

$G = \dfrac{E(r)}{2(1+\upsilon)}$, $E(r)$ and υ are the elastic constants.

Substituting for moment and shear resultants from equation (4.2.20) into equations (4.2.18) and (4.2.19), we get the following two equations of motion :

$$\frac{\partial D}{\partial r}\left(\frac{\partial \psi_r}{\partial r} + \frac{\upsilon}{r}\psi_r\right) + D\left(\frac{\partial^2 \psi_r}{\partial r^2} + \frac{1}{r}\frac{\partial \psi_r}{\partial r} - \frac{\psi_r}{r^2}\right) - \kappa G h\left(\frac{\partial w}{\partial r} + \psi_r\right) - \frac{\rho h^3}{12}\frac{\partial^2 \psi_r}{\partial t^2} = 0. \tag{4.2.21}$$

$$\kappa G h\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial \psi_r}{\partial r} + \frac{\psi_r}{r}\right) + \kappa\left(G\frac{\partial h}{\partial r} + h\frac{\partial G}{\partial r}\right)\left(\frac{\partial w}{\partial r} + \psi_r\right) - \rho h\frac{\partial^2 w}{\partial t^2} = 0 . \tag{4.2.22}$$

Introducing the non-dimensional variables

$$R = r/a, \quad H = h/a, \quad \overline{w} = w/a, \quad T = t\sqrt{E_0/\rho_0 a^2(1-\upsilon^2)}, \tag{4.2.23}$$

together with quadratic thickness variation, i.e.

$$H = h_0(1 + \alpha R + \beta R^2) \text{ such that } |\alpha| \le 1, |\beta| \le 1 \text{ and } \alpha + \beta > -1, \tag{4.2.24}$$

and assuming the exponential variation for the non-homogeneity of plate material as follows :

$$E = E_0 e^{\mu R}, \qquad \rho = \rho_0 e^{\eta R}, \tag{4.2.25}$$

equations (4.2.21) and (4.2.22) reduce to

$$\left(\mu H^2+3HH'\right)\left(R^2\frac{\partial\psi_r}{\partial R}+\upsilon R\psi_r\right)+H^2\left(R^2\frac{\partial^2\psi_r}{\partial R^2}+R\frac{\partial\psi_r}{\partial r}-\psi_r\right)-6\kappa(1-\upsilon)R^2\left(\frac{\partial\overline{w}}{\partial R}+\psi_r\right) \tag{4.2.26}$$
$$-R^2H^2e^{(\eta-\mu)R}\frac{\partial^2\psi_r}{\partial T^2}=0,$$

$$H\left(R\frac{\partial^2\overline{w}}{\partial R^2}+\frac{\partial\overline{w}}{\partial R}+R\frac{\partial\psi_r}{\partial R}+\psi_r\right)+R\left(H'+\mu H\right)\left(\frac{\partial\overline{w}}{\partial R}+\psi_r\right)-\frac{2}{\kappa(1-\upsilon)}RHe^{(\eta-\mu)R}\frac{\partial^2\overline{w}}{\partial T^2}=0, \tag{4.2.27}$$

where, μ and η are non-homogeneity parameters, α and β are the taper parameters, and h_0, ρ_0 and E_0 are the thickness, density and Young's modulus, respectively, at the centre of the plate.

For harmonic vibrations, the solution can be assumed as :

$$\overline{w}(R,T)=W(R)e^{i\Omega T} \quad \text{and} \quad \psi_r(R,T)=\psi(R)e^{i\Omega T}\ , \tag{4.2.28}$$

where, Ω is the frequency parameter.

Substitution of these solutions in equation (4.2.26) and (4.2.27) leads to

$$A_1\frac{dW}{dR}+A_2\frac{d^2\psi}{dR^2}+A_3\frac{d\psi}{dR}+(A_4+A_5\Omega^2)\psi=0, \tag{4.2.29}$$

$$B_1\frac{d^2W}{dR^2}+B_2\frac{dW}{dR}+B_3\Omega^2W+B_4\frac{d\psi}{dR}+B_5\psi=0, \tag{4.2.30}$$

where

$$\begin{aligned}
&A_1=-6\kappa(1-\upsilon)R^2, \quad A_2=H^2R^2, \quad A_3=(\mu H^2+3HH')R^2+H^2R,\\
&A_4=(\mu H^2+3HH')R\upsilon-H^2-6\kappa(1-\upsilon)R^2, \quad A_5=R^2H^2e^{(\eta-\mu)R},\\
&B_1=HR, \quad B_2=H(1+\mu R)+RH', \quad B_3=\frac{2}{\kappa(1-\upsilon)}RHe^{(\eta-\mu)R},\\
&B_4=B_1, \quad B_5=B_2, \quad \Omega=\omega\sqrt{\frac{\rho_0a^2(1-\upsilon^2)}{E_0}}.
\end{aligned} \tag{4.2.31}$$

Coupled differential equations (4.2.29) and (4.2.30) together with the edge conditions at $R = \varepsilon$ and $R = 1$, where $\varepsilon = b/a$, constitutes a well defined two point boundary value problem in the range (ε, 1), which has been solved by differential quadrature method.

3. METHOD OF SOLUTION : DQM

According to differential quadrature method (Bert et al.[1988]), equations (4.2.29) and (4.2.30) are discretized as follows :

$$\sum_{j=1}^{m} A_{1,i} c_{ij}^{(1)} W_j + \sum_{j=1}^{m} \left(A_{2,i} c_{ij}^{(2)} + A_{3,i} c_{ij}^{(1)} \right) \psi_j + \left(A_{4,i} + A_{5,i} \Omega^2 \right) \psi_i = 0 \quad , \tag{4.3.1}$$

$$\sum_{j=1}^{m} \left(B_{1,i} c_{ij}^{(2)} + B_{2,i} c_{ij}^{(1)} \right) W_j + B_{3,i} \Omega^2 W_i + \sum_{j=1}^{m} B_{4,i} c_{ij}^{(1)} \psi_j + B_{5,i} \psi_i = 0 \quad , \tag{4.3.2}$$

where $i = 2, 3, \ldots, m-1$. The weighting coefficients $c_{ij}^{(n)}$, for n^{th} order derivatives of W and ψ with respect to R, are determined by using relations (2.3.2)-(2.3.5).

The satisfaction of equations (4.3.1) and (4.3.2) at (m-2) nodal points x_i, $i = 2, 3, 4, \ldots, (m-1)$ provides a set of (2m-4) equations in terms of unknowns W_j, ψ_j, $j = 1, 2, \ldots, m$ (where W_j and ψ_j stand for $W(x_j)$ and $\psi(x_j)$, respectively). This can be written in matrix form as

$$[B][W^*] = [0] \quad , \tag{4.3.3}$$

where B and W^* are matrices of order (2m-4) x 2m and 2m x 1, respectively.

Here, the (m-2) internal grid points chosen for collocation are the zeros of shifted Chebyshev polynomial of order (m-2) with orthogonality range (ε,1) given by

$$x_{k+1} = \frac{1}{2}\left[(1+\varepsilon) + (1-\varepsilon)\cos\left(\frac{2k-1}{m-2}\frac{\pi}{2} \right) \right], \qquad k = 1, 2, \ldots, (m-2) \quad . \tag{4.3.4}$$

4. BOUNDARY CONDITIONS AND FREQUENCY EQUATIONS

The following three sets of boundary conditions have been considered:

(i) C-C : both the inner and outer edges clamped,

(ii) C-S : clamped at the inner edge and simply supported at the outer, and

(iii) C-F : clamped at the inner edge and free at the outer.

The relations which should be satisfied at a clamped, simply supported and free edge are

$$W = \psi = 0; \tag{4.4.1}$$

$$W = \frac{\partial \psi}{\partial R} + \frac{\upsilon}{R}\psi = 0; \tag{4.4.2}$$

$$\psi + \frac{\partial W}{\partial R} = \frac{\partial \psi}{\partial R} + \frac{\upsilon}{R}\psi = 0 , \tag{4.4.3}$$

respectively.

Discretization of relations (4.4.1)-(4.4.3) on two edges of the plate leads to

$$\text{(C-C)} \quad W_1 = 0; \quad \psi_1 = 0 ; \quad W_m = 0 ; \quad \psi_m = 0 , \tag{4.4.4}$$

$$\text{(C-S)} \quad W_1 = 0 ; \quad \psi_1 = 0 ; \quad W_m = 0 ; \quad \sum_{j=1}^{m} c_{mj}^{(1)} \psi_j + \upsilon \psi_m = 0 \text{, and} \tag{4.4.5}$$

$$\text{(C-F)} \quad W_1 = 0 ; \quad \psi_1 = 0 ; \quad \psi_m + \sum_{j=1}^{m} c_{mj}^{(1)} W_j = 0 ; \quad \sum_{j=1}^{m} c_{mj}^{(1)} \psi_j + \upsilon \psi_m = 0 , \tag{4.4.6}$$

which gives a set of four homogeneous equations (4.4.4), (4.4.5) and (4.4.6). For a C-C plate, these equations together with the field equations (4.3.3) yield a complete set of $2m$ equations in $2m$ unknowns, which can be written as

$$\begin{bmatrix} B \\ B^{CC} \end{bmatrix} [W^*] = [0], \tag{4.4.7}$$

where B^{CC} is a matrix of order 4 x $2m$.

For a nontrivial solution of equation (4.4.7), the frequency determinant is given by,

$$\begin{vmatrix} B \\ B^{CC} \end{vmatrix} = 0 . \tag{4.4.8}$$

Similarly for C-S and C-F plates, frequency determinants can be written as

$$\begin{vmatrix} B \\ B^{CS} \end{vmatrix} = 0 , \quad \begin{vmatrix} B \\ B^{CF} \end{vmatrix} = 0 , \qquad (4.4.9, 4.4.10)$$

respectively.

5. NUMERICAL RESULTS AND DISCUSSION

The values of the frequency parameter Ω have been obtained by solving equations (4.4.8)-(4.4.10) for various values of plate parameters. Numerical results have been computed for the first three modes of vibration to investigate the effect of non-homogeneity parameter μ = -0.5(0.1)1.0, density parameter η = -0.5(0.1)1.0 and thickness parameter h_0 = 0.03, 0.05(0.025)0.2 and taper parameters α = -0.5(0.1)0.5, β = -0.5(0.1)0.5 such that $\alpha + \beta > -1$, on the natural frequencies for two radii ratios ε = 0.3, 0.5 by Shear Plate Theory of Mindlin(SPT) and Classical Plate Theory(CPT) for $\upsilon = 0.3$. For determining the results on the basis of classical plate theory, the governing equation of motion is obtained by eliminating Q_r from equations (4.2.18) and (4.2.19) after neglecting the rotatory inertia term in equation (4.2.18) and then substituting $\psi_r = -\dfrac{\partial w}{\partial r}$ in the resulting equation. The averaging shear constant is taken to be $\dfrac{\pi^2}{12}$.

To choose appropriate value of the number of collocation points m, the computer program developed for the evaluation of the frequency parameter Ω was run for m = 8(1)20 for different sets of plate parameters for the three sets of boundary conditions. Figures 4.1(a.b.c) show the convergence of first three frequency parameters with the number of collocation points m for $h_0 = 0.1$, $\mu = 1.0$, $\eta = -0.5$, $\alpha = 0.0$, $\beta = 0.5$ and $\varepsilon = 0.3$ for C-C, C-S and C-F plates respectively. It is observed that four digit exactitude in values of frequency parameter Ω can be attained by fixing $m = 14$(Calculations were carried out with double precision arithmetic.) remove

The results have been given in Tables (4.1-4.18) and Figures (4.2-4.8). Tables (4.1-4.18) present the frequency parameter obtained by CPT(Ω_c) for h_0 = 0.1 and SPT(Ω_s) for h_0 = 0.05, 0.1, 0.2 taking various values of non-homogeneity parameter μ = -0.5, 0.0, 1.0, density parameter η = -0.5, 0.0, 1.0 and radii ratio ε = 0.3, 0.5 for C-C, C-S and C-F plates. In the case of classical theory, h_0 does not appear explicitly in the governing differential equation except in the final expression of Ω. Therefore, the frequencies are computed for general value of h_0 and then transformed to the required value of $h_0 = 0.1$ by using a multiplying factor $h_0/\sqrt{12}$. From the tables it is found that the frequencies for C-S plate are higher than C-F plate and lower than the C-C plate for the same set of values of other plate parameters. The frequency parameter increases with the increase in the radii ratio ε, thickness parameter h_0, non-homogeneity parameter μ and taper parameters α as well as β, while it decreases with increase in density parameter η.

Figures 4.2(a,b,c) show the effect of non-homogeneity parameter μ on the frequency parameter Ω for all the three boundary conditions for the first three modes of vibration for radii ratio $\varepsilon = 0.3$ and fixed values of density parameter $\eta = -0.5$, taper parameters $\alpha = 0.5$; $\beta = 0.5$ with

two values of thickness parameter $h_0 = 0.05, 0.1$. It is observed that the frequency parameter Ω increases with increasing values of non-homogeneity parameter μ, whatever be the other plate parameters. The rate of increase of frequency parameter Ω with non-homogeneity parameter μ is higher in the third mode as compared to those in both the fundamental and second modes. The effect of transverse shear and rotatory inertia increases with the increasing values of non-homogeneity parameter μ. It also increases with the increase in the number of modes.

Figures 4.3(a,b,c) show the effect of density parameter η on the frequency parameter Ω for all the three boundary conditions for the first three modes of vibration for radii ratio $\varepsilon = 0.3$ and non-homogeneity parameter $\mu = 1.0$, taper parameters $\alpha = 0.5$; $\beta = 0.5$ with two values of thickness parameter $h_0 = 0.05, 0.1$. It is seen that the frequency parameter Ω decreases with the increasing values of the density parameter η keeping all other plate parameters fixed. The rate of decrease of frequency parameter Ω with increasing values of density parameter η increases with the increase in the number of modes.

Figures 4.4(a,b,c) depict the variation of frequency parameter Ω with taper parameter α for the first three modes of vibration for radii ratio $\varepsilon = 0.3$, non-homogeneity parameter $\mu = 1.0$, density parameter $\eta = -0.5$, taper parameter $\beta = 0.5$ and thickness parameter $h_0 = 0.05, 0.1$ for all three plates. It is observed that frequency parameter increases with increasing values of taper parameter α. The increase is more pronounced in the case of C-C plate as compared to C-S and C-F plates. The effect of transverse shear and rotatory inertia becomes significant with increasing values of α and also the number of modes. Figures 4.5(a,b,c) show the plots of frequency parameter Ω versus taper parameter β for radii ratio $\varepsilon = 0.3$ and fixed values of non-homogeneity parameter $\mu = 1.0$, density parameter $\eta = -0.5$, taper parameter $\alpha = 0.5$ with

thickness parameter h_0 = 0.05, 0.1. It is observed that frequency parameter increases with the increasing values of taper parameter β. The increase is more pronounced in case of C-C plate as compared to C-S and C-F plates. The rate of increase of frequency parameter Ω with taper parameter β increases with increasing order of modes.

The variation of frequency parameter Ω with radii ratio ε for all the three modes and boundary conditions for non-homogeneity parameter μ = 1.0 and density parameter η = -0.5, taper parameters α = 0.5; β = 0.5, thickness parameter h_0 = 0.05, 0.1 has been shown in Figures 4.6(a,b,c). It is found that the frequency parameter can be increased /decreased by increasing/decreasing the hole size. The increase of frequency parameter Ω is more pronounced for $\varepsilon > 0.5$ as compared to that for $\varepsilon < 0.5$. The effect of transverse shear and rotatory inertia increases with increasing hole size for all the three plates.

Figures 4.7(a,b,c) show the effect of thickness parameter h_0 on frequency parameters Ω_s *and* Ω_c for radii ratio ε = 0.3, 0.5 for all the three plates for μ = 1.0, η = -0.5, α = 0.5 and β = 0.5. It is observed that the effect of transverse shear and rotatory inertia increases with increasing value of thickness parameter h_0. It also increases with the increasing value of radii ratio ε. The effect of transverse shear and rotatory inertia has been found to increase with the increase in the number of modes.

Normalized displacements have been plotted in Figures 4.8(a,b,c) for the first three modes of vibration for ε = 0.3, η = -0.5, μ = 1.0, h_0 = 0.1 and three combinations of taper parameters α = 0.0; β = 0.0; α = 0.5; β = 0.0; α = 0.5; β = 0.5. These figures show that the nodal circles shift towards the inner edge as the plate becomes thicker and thicker towards the outer edge. A

comparison of results for homogeneous ($\mu = 0.0$, $\eta = 0.0$) uniform thickness ($\alpha = 0.0$, $\beta = 0.0$) Mindlin's annular plates has been presented in Table 4.19 with analytical solutions given by Irie et al.[1982]. A close agreement of the results shows the versatility of the present technique.

From the above discussion it can be concluded that the effects of transverse shear and rotatory inertia cannot be neglected while dealing with vibration of non-homogeneous moderately thick ($h_0 > 0.1$) plates. A similar inference was obtained by Deresiewicz and Mindlin [1955] for isotropic homogeneous circular disks.

Table 4.1

Values of frequency parameter Ω for C-C plate for η = -0.5, ε = 0.3

Mode	α	β	μ = -0.5					μ = 0.0					μ = 1.0				
			*	Ωc	Ωs			*	Ωc	Ωs			*	Ωc	Ωs		
				h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2
I	-0.5	0	29.8639	0.8621	0.4233	0.8051	1.3713	35.1222	1.0139	0.4977	0.9459	1.6081	48.5495	1.4015	0.6874	1.3033	2.2026
		0.5	41.9983	1.2124	0.5866	1.0758	1.6867	49.5019	1.4290	0.6913	1.2672	1.9860	68.7403	1.9844	0.9590	1.7543	2.7406
	0	-0.5	32.8105	0.9472	0.4630	0.8704	1.4405	38.5799	1.1137	0.5442	1.0221	1.6876	53.2999	1.5386	0.7509	1.4060	2.3058
		0	45.2406	1.3060	0.6279	1.1350	1.7305	53.2976	1.5386	0.7395	1.3359	2.0355	73.9349	2.1343	1.0245	1.8458	2.8025
		0.5	56.5277	1.6318	0.7701	1.3390	1.9123	66.6883	1.9251	0.9085	1.5794	2.2575	92.7847	2.6785	1.2625	2.1907	3.1301
	0.5	-0.5	48.4295	1.3980	0.6677	1.1893	1.7666	57.0322	1.6464	0.7859	1.3988	2.0760	79.0459	2.2819	1.0875	1.9291	2.8525
		0	59.9148	1.7296	0.8096	1.3855	1.9338	70.6509	2.0395	0.9545	1.6330	2.2810	98.2027	2.8349	1.3247	2.2610	3.1573
		0.5	70.8150	2.0443	0.9364	1.5421	2.0477	83.5909	2.4131	1.1054	1.8212	2.4227	116.4417	3.3614	1.5379	2.5305	3.3742
II	-0.5	0	82.8038	2.3903	1.1489	2.0762	3.1701	97.5741	2.8167	1.3530	2.4415	3.7203	135.1833	3.9024	1.8709	3.3624	5.0960
		0.5	114.7703	3.3131	1.5437	2.6207	3.6262	135.0557	3.8987	1.8163	3.0835	4.2700	186.5973	5.3866	2.5064	4.2474	5.8855
	0	-0.5	92.2704	2.6636	1.2676	2.2424	3.3003	108.7509	3.1394	1.4926	2.6356	3.8709	150.7274	4.3511	2.0635	3.6253	5.2975
		0	125.2016	3.6143	1.6617	2.7561	3.7018	147.3683	4.2542	1.9553	3.2419	4.3576	203.7097	5.8806	2.6979	4.4626	6.0052
		0.5	154.6659	4.4648	1.9781	3.0992	3.9157	181.9028	5.2511	2.3277	3.6508	4.6212	251.0422	7.2470	3.2105	5.0369	6.4011
	0.5	-0.5	135.4076	3.9089	1.7723	2.8738	3.7601	159.4160	4.6019	2.0853	3.3793	4.4251	220.4582	6.3641	2.8768	4.6485	6.0981
		0	165.5379	4.7787	2.0829	3.1925	3.9514	194.7346	5.6215	2.4510	3.7600	4.6624	268.8736	7.7617	3.3803	5.1853	6.4585
		0.5	193.8493	5.5959	2.3413	3.4190	4.0753	227.9109	6.5792	2.7564	4.0330	4.8165	314.3244	9.0738	3.8031	5.5769	6.6940
III	-0.5	0	162.8812	4.7020	2.1999	3.7698	5.3286	192.0798	5.5449	2.5916	4.4331	6.2598	266.3226	7.6881	3.5825	6.0980	8.5876
		0.5	224.2744	6.4742	2.8815	4.5591	5.8809	263.7249	7.6131	3.3883	5.3617	6.9211	363.6012	10.4963	4.6643	7.3713	9.5299
	0	-0.5	182.6486	5.2726	2.4274	4.0452	5.5266	215.4837	6.2205	2.8596	4.7553	6.4918	299.0142	8.6318	3.9525	6.5352	8.9058
		0	245.9753	7.1007	3.0956	4.7670	6.0038	289.3880	8.3539	3.6407	5.6055	7.0649	399.3650	11.5287	5.0123	7.7040	9.7271
		0.5	302.3385	8.7278	3.5981	5.2072	6.2609	355.1183	10.2514	4.2304	6.1291	7.3722	488.4935	14.1016	5.8182	8.4379	10.1662
	0.5	-0.5	267.2234	7.7141	3.2924	4.9446	6.1046	314.5232	9.0795	3.8724	5.8136	7.1828	434.4141	12.5405	5.3310	7.9874	9.8888
		0	324.9222	9.3797	3.7756	5.3448	6.3331	381.8152	11.0221	4.4397	6.2902	7.4564	525.6695	15.1748	6.1065	8.6580	10.2804
		0.5	378.9667	10.9398	4.1549	5.6184	6.4884	444.8192	12.8408	4.8873	6.6171	7.6406	611.0422	17.6393	6.7237	9.1220	10.5390

* for general value of h_0

Table 4.2

Values of frequency parameter Ω for C-C plate for $\eta = 0.0$, $\varepsilon = 0.3$

Mode	α	β	$\mu = -0.5$					$\mu = 0.0$					$\mu = 1.0$				
			*	Ωc	Ωs			*	Ωc	Ωs			*	Ωc	Ωs		
				$h_0=0.1$	$h_0=0.05$	$h_0=0.1$	$h_0=0.2$		$h_0=0.1$	$h_0=0.05$	$h_0=0.1$	$h_0=0.2$		$h_0=0.1$	$h_0=0.05$	$h_0=0.1$	$h_0=0.2$
I	-0.5	0	25.2335	0.7284	0.3578	0.6808	1.1611	29.7276	0.8582	0.4214	0.8014	1.3648	41.2340	1.1903	0.5841	1.1085	1.8784
		0.5	35.7121	1.0309	0.4988	0.9147	1.4340	42.1662	1.2172	0.5889	1.0798	1.6930	58.7592	1.6962	0.8200	1.5012	2.3491
	0	-0.5	27.6949	0.7995	0.3909	0.7354	1.2192	32.6202	0.9417	0.4603	0.8653	1.4319	45.2199	1.3054	0.6374	1.1951	1.9661
		0	38.4248	1.1092	0.5333	0.9642	1.4704	45.3462	1.3090	0.6293	1.1373	1.7344	63.1230	1.8222	0.8751	1.5783	2.4011
		0.5	48.1855	1.3910	0.6563	1.1403	1.6268	56.9460	1.6439	0.7757	1.3483	1.9262	79.5089	2.2952	1.0822	1.8790	2.6868
	0.5	-0.5	41.0922	1.1862	0.5666	1.0096	1.5004	48.4746	1.3993	0.6682	1.1901	1.7683	67.4160	1.9461	0.9281	1.6485	2.4432
		0	51.0210	1.4728	0.6893	1.1791	1.6445	60.2677	1.7398	0.8142	1.3931	1.9456	84.0623	2.4267	1.1344	1.9380	2.7092
		0.5	60.4540	1.7452	0.7989	1.3140	1.7419	71.4856	2.0636	0.9450	1.5558	2.0674	99.9300	2.8847	1.3201	2.1728	2.8975
II	-0.5	0	70.0158	2.0212	0.9719	1.7582	2.6891	82.6289	2.3853	1.1465	2.0717	3.1639	114.8228	3.3146	1.5906	2.8646	4.3557
		0.5	97.5812	2.8169	1.3122	2.2267	3.0781	114.9999	3.3198	1.5467	2.6258	3.6346	159.3617	4.6004	2.1419	3.6334	5.0373
	0	-0.5	77.9522	2.2503	1.0716	1.8984	2.7996	92.0144	2.6562	1.2640	2.2359	3.2921	127.9184	3.6927	1.7534	3.0885	4.5283
		0	106.3456	3.0699	1.4114	2.3405	3.1417	125.3621	3.6189	1.6637	2.7595	3.7084	173.8117	5.0175	2.3040	3.8164	5.1385
		0.5	131.7859	3.8043	1.6837	2.6337	3.3222	155.2258	4.4810	1.9851	3.1104	3.9316	214.8658	6.2026	2.7487	4.3131	5.4762
	0.5	-0.5	114.9162	3.3173	1.5043	2.4396	3.1907	135.4958	3.9114	1.7733	2.8757	3.7652	187.9462	5.4255	2.4556	3.9747	5.2167
		0	140.9273	4.0682	1.7716	2.7119	3.3521	166.0329	4.7930	2.0890	3.2023	3.9659	229.9343	6.6376	2.8925	4.4390	5.5239
		0.5	165.3884	4.7744	1.9935	2.9041	3.4566	194.7409	5.6217	2.3519	3.4349	4.0961	269.3815	7.7764	3.2588	4.7757	5.7242
III	-0.5	0	137.7694	3.9771	1.8621	3.1953	4.5216	162.6951	4.6966	2.1973	3.7654	5.3248	226.2176	6.5303	3.0478	5.2016	7.3405
		0.5	190.6768	5.5044	2.4488	3.8717	4.9910	224.5293	6.4816	2.8847	4.5641	5.8881	310.4221	8.9611	3.9855	6.3044	8.1477
	0	-0.5	154.3641	4.4561	2.0537	3.4285	4.6894	182.3743	5.2647	2.4237	4.0392	5.5219	253.7956	7.3264	3.3617	5.5756	7.6119
		0	208.9360	6.0315	2.6291	4.0472	5.0949	246.1573	7.1059	3.0978	4.7706	6.0098	340.6606	9.8340	4.2811	6.5880	8.3150
		0.5	257.5676	7.4353	3.0600	4.4214	5.3141	302.9540	8.7455	3.6052	5.2172	6.2720	417.8973	12.0637	4.9786	7.2185	8.6903
	0.5	-0.5	226.8042	6.5473	2.7949	4.1973	5.1800	267.3284	7.7171	3.2936	4.9470	6.1095	370.2780	10.6890	4.5521	6.8298	8.4521
		0	276.5837	7.9843	3.2093	4.5375	5.3750	325.4702	9.3955	3.7818	5.3535	6.3431	449.3538	12.9717	5.2235	7.4057	8.7871
		0.5	323.2438	9.3312	3.5331	4.7696	5.5081	379.9443	10.9680	4.1653	5.6313	6.5013	523.3785	15.1086	5.7560	7.8022	9.0096

* for general value of h_0

Table 4.3

Values of frequency parameter Ω for C-C plate for $\eta = 1.0, \varepsilon = 0.3$

Mode	α	β	μ = -0.5					μ = 0.0					μ = 1.0				
			*	Ωc	Ωs			*	Ωc	Ωs			*	Ωc	Ωs		
				h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2
I	-0.5	0	17.9224	0.5174	0.2542	0.4840	0.8268	21.1864	0.6116	0.3004	0.5719	0.9765	29.5887	0.8542	0.4194	0.7972	1.3569
		0.5	25.6880	0.7415	0.3587	0.6573	1.0287	30.4364	0.8786	0.4250	0.7792	1.2211	42.7120	1.2330	0.5963	1.0926	1.7129
	0	-0.5	19.6296	0.5667	0.2772	0.5219	0.8673	23.1986	0.6697	0.3275	0.6165	1.0236	32.3772	0.9346	0.4568	0.8582	1.4194
		0	27.5750	0.7960	0.3827	0.6915	1.0535	32.6543	0.9426	0.4532	0.8192	1.2495	45.7712	1.3213	0.6349	1.1467	1.7491
		0.5	34.8303	1.0055	0.4739	0.8216	1.1677	41.3069	1.1924	0.5623	0.9760	1.3909	58.0809	1.6766	0.7906	1.3731	1.9635
	0.5	-0.5	29.4297	0.8496	0.4057	0.7229	1.0739	34.8354	1.0056	0.4803	0.8559	1.2728	48.7804	1.4082	0.6721	1.1961	1.7785
		0	36.8046	1.0625	0.4968	0.8481	1.1793	43.6256	1.2594	0.5891	1.0069	1.4036	61.2751	1.7689	0.8272	1.4142	1.9782
		0.5	43.8273	1.2652	0.5780	0.9472	1.2494	52.0067	1.5013	0.6865	1.1273	1.4922	73.2157	2.1136	0.9669	1.5906	2.1179
II	-0.5	0	49.8348	1.4386	0.6921	1.2534	1.9203	58.9884	1.7028	0.8191	1.4828	2.2708	82.4649	2.3806	1.1440	2.0668	3.1581
		0.5	70.2255	2.0272	0.9432	1.5967	2.1995	83.0083	2.3962	1.1157	1.8915	2.6114	115.7177	3.3405	1.5560	2.6412	3.6595
	0	-0.5	55.3841	1.5988	0.7620	1.3522	1.9989	65.5721	1.8929	0.9019	1.5993	2.3626	91.7111	2.6475	1.2596	2.2276	3.2832
		0	76.3799	2.2049	1.0127	1.6763	2.2439	90.3086	2.6070	1.1981	1.9857	2.6631	125.9659	3.6363	1.6713	2.7720	3.7308
		0.5	95.2498	2.7496	1.2128	1.8876	2.3710	112.5280	3.2484	1.4356	2.2407	2.8211	156.6985	4.5235	2.0033	3.1391	3.9734
	0.5	-0.5	82.3916	2.3784	1.0778	1.7457	2.2783	97.4400	2.8128	1.2753	2.0676	2.7029	135.9786	3.9254	1.7792	2.8854	3.7856
		0	101.6775	2.9352	1.2742	1.9419	2.3917	120.1522	3.4685	1.5085	2.3049	2.8448	167.4004	4.8324	2.1055	3.2284	4.0059
		0.5	119.8468	3.4597	1.4361	2.0784	2.4656	141.5418	4.0860	1.7016	2.4716	2.9371	196.9706	5.6861	2.3779	3.4739	4.1492
III	-0.5	0	98.1500	2.8333	1.3276	2.2812	3.2312	116.2328	3.3554	1.5718	2.6996	3.8240	162.5270	4.6918	2.1948	3.7609	5.3235
		0.5	137.2525	3.9621	1.7589	2.7726	3.5676	162.0701	4.6786	2.0796	3.2839	4.2292	225.3184	6.5044	2.8943	4.5791	5.9098
	0	-0.5	109.7892	3.1693	1.4626	2.4468	3.3507	130.0796	3.7551	1.7320	2.8953	3.9649	182.0572	5.2555	2.4191	4.0318	5.5192
		0	150.1161	4.3335	1.8856	2.8962	3.6413	177.3563	5.1198	2.2303	3.4303	4.3158	246.8294	7.1254	3.1057	4.7831	6.0294
		0.5	186.1510	5.3737	2.1996	3.1639	3.8001	219.5657	6.3383	2.6023	3.7517	4.5059	304.5642	8.7920	3.6234	5.2431	6.3022
	0.5	-0.5	162.6901	4.6965	2.0022	3.0022	3.7017	192.3026	5.5513	2.3688	3.5557	4.3867	267.8748	7.7329	3.2997	4.9572	6.1274
		0	199.5669	5.7610	2.3042	3.2456	3.8431	235.5025	6.7984	2.7267	3.8482	4.5563	326.9747	9.4389	3.7983	5.3770	6.3714
		0.5	234.1865	6.7604	2.5376	3.4113	3.9404	276.0382	7.9685	3.0052	4.0473	4.6724	382.3769	11.0383	4.1905	5.6637	6.5355

* for general value of h_0

Table 4.4

Values of frequency parameter Ω for C-C plate for $\eta = -0.5$, $\varepsilon = 0.5$

Mode	α	β	μ = -0.5					μ = 0.0					μ = 1.0				
			*	Ωc	Ωs			*	Ωc	Ωs			*	Ωc	Ωs		
				h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2
I	-0.5	0	55.0406	1.5889	0.7716	1.4277	2.2796	66.3711	1.9160	0.9303	1.7203	2.7441	96.4768	2.7850	1.3513	2.4948	3.9659
		0.5	82.6856	2.3869	1.1223	1.9370	2.7362	99.8274	2.8818	1.3548	2.3381	3.3032	145.4660	4.1992	1.9728	3.4001	4.7996
	0	-0.5	60.8822	1.7575	0.8475	1.5421	2.3801	73.3962	2.1188	1.0214	1.8571	2.8625	106.6310	3.0782	1.4824	2.6895	4.1289
		0	89.1591	2.5738	1.1976	2.0269	2.7859	107.6079	3.1064	1.4451	2.4450	3.3604	156.6965	4.5234	2.1021	3.5503	4.8744
		0.5	115.8277	3.3437	1.4917	2.3576	2.9928	139.8897	4.0383	1.8016	2.8480	3.6183	203.9925	5.8888	2.6248	4.1456	5.2731
	0.5	-0.5	95.4742	2.7561	1.2686	2.1063	2.8249	115.1996	3.3255	1.5302	2.5393	3.4051	167.6585	4.8399	2.2238	3.6822	4.9320
		0	122.5308	3.5372	1.5572	2.4157	3.0120	147.9452	4.2708	1.8800	2.9165	3.6391	215.6169	6.2243	2.7365	4.2399	5.2967
		0.5	148.7924	4.2953	1.8014	2.6325	3.1207	179.7381	5.1886	2.1764	3.1824	3.7770	262.2092	7.5693	3.1719	4.6375	5.5178
II	-0.5	0	152.1449	4.3920	2.0600	3.5410	4.9831	183.6653	5.3020	2.4855	4.2681	6.0009	267.3373	7.7174	3.6123	6.1856	8.6771
		0.5	226.8497	6.5486	2.8759	4.4518	5.5541	273.6121	7.8985	3.4690	5.3712	6.7055	397.5796	11.4771	5.0369	7.7953	9.7484
	0	-0.5	169.6359	4.8970	2.2622	3.7825	5.1312	204.8257	5.9128	2.7292	4.5569	6.1755	298.2691	8.6103	3.9654	6.5968	8.9211
		0	246.2130	7.1076	3.0577	4.6100	5.6179	297.0313	8.5746	3.6880	5.5599	6.7795	431.7954	12.4649	5.3535	8.0628	9.8501
		0.5	318.0874	9.1824	3.6542	5.0793	5.8399	383.5626	11.0725	4.4093	6.1337	7.0577	557.0690	16.0812	6.4032	8.9171	10.2878
	0.5	-0.5	265.0816	7.6522	3.2236	4.7425	5.6654	319.8538	9.2334	3.8876	5.7175	6.8341	465.1442	13.4276	5.6413	8.2853	9.9248
		0	338.1189	9.7607	3.7932	5.1660	5.8611	407.7882	11.7718	4.5763	6.2363	7.0810	592.4579	17.1028	6.6436	9.0611	10.3179
		0.5	408.8108	11.8014	4.2254	5.4299	5.9817	492.8926	14.2286	5.1010	6.5616	7.2336	715.6553	20.6592	7.4125	9.5545	10.5611
III	-0.5	0	298.6657	8.6217	3.8758	6.2213	8.1416	360.7037	10.4126	4.6769	7.4992	9.8128	525.3153	15.1645	6.7951	10.8635	14.2129
		0.5	443.8486	12.8128	5.2105	7.4418	8.8581	535.1213	15.4476	6.2831	8.9762	10.6864	776.6651	22.4204	9.1118	13.0175	15.5112
	0	-0.5	334.1550	9.6462	4.2339	6.5784	8.3584	403.7500	11.6553	5.1091	7.9272	10.0732	588.5360	16.9896	7.4221	11.4761	14.5883
		0	483.0798	13.9453	5.5042	7.6601	8.9723	582.6704	16.8202	6.6373	9.2377	10.8229	846.3941	24.4333	9.6246	13.3919	15.7062
		0.5	622.6037	17.9730	6.3697	8.2400	9.2855	750.2640	21.6583	7.6832	9.9427	11.2028	1087.8383	31.4032	11.1438	14.4329	16.2637
	0.5	-0.5	521.3217	15.0493	5.7660	7.8427	9.0668	629.0301	18.1585	6.9526	9.4562	10.9359	914.4073	26.3967	10.0797	13.7049	15.8673
		0	663.1699	19.1441	6.5749	8.3630	9.3457	799.4212	23.0773	7.9300	10.0896	11.2746	1159.8965	33.4833	11.4997	14.6426	16.3648
		0.5	800.3295	23.1035	7.1487	8.6968	8.9527	964.1640	27.8330	8.6257	10.4947	10.8427	1397.2028	40.3338	12.5177	15.2400	15.8226

* for general value of h_0

Table 4.5

Values of frequency parameter Ω for C-C plate for $\eta = 0.0$, $\varepsilon = 0.5$

Mode	α	β	μ = -0.5					μ = 0.0					μ = 1.0				
			*	Ωc	Ωs			*	Ωc	Ωs			*	Ωc	Ωs		
				h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2
I	-0.5	0	45.4803	1.3129	0.6377	1.1803	1.8862	54.8912	1.5846	0.7695	1.4238	2.2736	79.9310	2.3074	1.1199	2.0691	3.2949
		0.5	68.5852	1.9799	0.9308	1.6063	2.2682	82.8778	2.3925	1.1248	1.9412	2.7425	120.9834	3.4925	1.6411	2.8298	3.9973
	0	-0.5	50.2552	1.4507	0.6997	1.2740	1.9686	60.6381	1.7505	0.8441	1.5359	2.3710	88.2508	2.5476	1.2274	2.2292	3.4295
		0	73.8837	2.1328	0.9924	1.6797	2.3086	89.2508	2.5764	1.1987	2.0287	2.7892	130.1962	3.7584	1.7473	2.9532	4.0586
		0.5	96.1804	2.7765	1.2382	1.9558	2.4809	116.2646	3.3563	1.4971	2.3659	3.0045	169.8450	4.9030	2.1858	3.4534	4.3933
	0.5	-0.5	79.0509	2.2820	1.0505	1.7446	2.3404	95.4673	2.7559	1.2684	2.1060	2.8257	139.1864	4.0180	1.8472	3.0617	4.1058
		0	101.6689	2.9349	1.2918	2.0032	2.4964	122.8654	3.5468	1.5613	2.4219	3.0213	179.3843	5.1784	2.2775	3.5308	4.4123
		0.5	123.6288	3.5689	1.4956	2.1838	2.5866	149.4744	4.3150	1.8092	2.6439	3.1359	218.4497	6.3061	2.6429	3.8644	4.5971
II	-0.5	0	125.7716	3.6307	1.7036	2.9305	4.1275	151.9449	4.3863	2.0573	3.5362	4.9775	221.5067	6.3943	2.9953	5.1366	7.2171
		0.5	188.1494	5.4314	2.3845	3.6892	4.6002	227.1080	6.5560	2.8791	4.4569	5.5617	330.5110	9.5410	4.1889	6.4855	8.1082
	0	-0.5	140.1050	4.0445	1.8696	3.1296	4.2503	169.2995	4.8873	2.2576	3.7749	5.1224	246.9172	7.1279	3.2863	5.4777	7.4201
		0	204.0397	5.8901	2.5337	3.8194	4.6529	246.3428	7.1113	3.0592	4.6126	5.6227	358.6601	10.3536	4.4501	6.7071	8.1920
		0.5	264.0721	7.6231	3.0305	4.2080	4.8362	318.6732	9.1993	3.6609	5.0889	5.8525	463.5359	13.3811	5.3287	7.4191	8.5539
	0.5	-0.5	219.5181	6.3369	2.6701	3.9287	4.6923	265.0800	7.6522	3.2235	4.7428	5.6678	386.0850	11.1453	4.6877	6.8916	8.2532
		0	280.5172	8.0978	3.1445	4.2795	4.8538	338.5780	9.7739	3.7983	5.1735	5.8716	492.6616	14.2219	5.5272	7.5383	8.5782
		0.5	339.5698	9.8025	3.5034	4.4974	4.9537	409.7249	11.8277	4.2348	5.4424	5.9981	595.8123	17.1996	6.1695	7.9475	8.7801
III	-0.5	0	246.9403	7.1286	3.2065	5.1514	6.7439	298.4481	8.6155	3.8729	6.2170	8.1385	435.2742	12.5653	5.6374	9.0274	11.8173
		0.5	368.1161	10.6266	4.3190	6.1655	7.3382	444.1320	12.8210	5.2137	7.4464	8.8633	645.5241	18.6347	7.5771	10.8266	12.8960
	0	-0.5	276.0525	7.9689	3.5013	5.4469	6.9231	333.7878	9.6356	4.2292	6.5717	8.3537	487.2601	14.0660	6.1558	9.5369	12.1280
		0	400.3433	11.5569	4.5607	6.3459	7.4323	483.2237	13.9495	5.5057	7.6627	8.9760	702.9437	20.2922	8.0014	11.1371	13.0571
		0.5	516.8282	14.9195	5.2797	6.8264	7.6931	623.2453	17.9915	6.3760	8.2473	9.2926	904.9586	26.1239	9.2703	12.0021	13.5226
	0.5	-0.5	431.7455	12.4634	4.7764	6.4969	7.5102	521.3225	15.0493	5.7659	7.8436	9.0692	758.9282	21.9084	8.3783	11.3967	13.1902
		0	550.1646	15.8819	5.4486	6.9281	7.7424	663.6744	19.1586	6.5797	8.3688	9.3516	964.3190	27.8375	9.5649	12.1758	13.6062
		0.5	664.6893	19.1879	5.9239	7.2051	7.4032	801.3302	23.1324	7.1570	8.7052	8.9868	1162.8933	33.5698	10.4127	12.6721	13.1756

* for general value of h_0

Table 4.6

Values of frequency parameter Ω for C-C plate for $\eta = 1.0$, $\varepsilon = 0.5$

Mode	α	β	μ = -0.5					μ = 0.0					μ = 1.0				
			*	Ωc	Ωs			*	Ωc	Ωs			*	Ωc	Ωs		
				h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2
I	-0.5	0	30.9707	0.8940	0.4343	0.8043	1.2866	37.4451	1.0809	0.5251	0.9722	1.5550	54.7192	1.5796	0.7671	1.4189	2.2657
		0.5	47.0626	1.3586	0.6384	1.1007	1.5520	56.9715	1.6446	0.7730	1.3335	1.8825	83.4640	2.4094	1.1324	1.9535	2.7610
	0	-0.5	34.1511	0.9859	0.4757	0.8667	1.3416	41.2789	1.1916	0.5749	1.0472	1.6203	60.2866	1.7403	0.8391	1.5266	2.3568
		0	50.6006	1.4607	0.6795	1.1495	1.5786	61.2334	1.7677	0.8224	1.3918	1.9132	89.6431	2.5878	1.2036	2.0362	2.8016
		0.5	66.1417	1.9093	0.8505	1.3408	1.6971	80.0961	2.3122	1.0305	1.6265	2.0621	117.4281	3.3899	1.5112	2.3873	3.0355
	0.5	-0.5	54.0490	1.5603	0.7182	1.1926	1.5995	65.3883	1.8876	0.8690	1.4433	1.9372	95.6699	2.7618	1.2707	2.1091	2.8330
		0	69.8092	2.0152	0.8861	1.3720	1.7070	84.5134	2.4397	1.0733	1.6635	2.0728	123.8304	3.5747	1.5726	2.4389	3.0476
		0.5	85.1208	2.4572	1.0276	1.4965	1.7685	103.0999	2.9762	1.2460	1.8173	2.1514	151.2170	4.3653	1.8287	2.6724	3.1756
II	-0.5	0	85.7497	2.4754	1.1620	2.0004	2.8199	103.7533	2.9951	1.4057	2.4194	3.4100	151.7182	4.3797	2.0540	3.5303	4.9719
		0.5	129.1321	3.7277	1.6343	2.5235	3.1418	156.1094	4.5065	1.9773	3.0568	3.8090	227.8836	6.5784	2.8885	4.4717	5.5844
	0	-0.5	95.3497	2.7525	1.2734	2.1350	2.9037	115.3957	3.3312	1.5406	2.5813	3.5093	168.8209	4.8734	2.2510	3.7635	5.1114
		0	139.8044	4.0358	1.7344	2.6112	3.1777	169.0492	4.8800	2.0985	3.1620	3.8505	246.8832	7.1269	3.0654	4.6226	5.6409
		0.5	181.5828	5.2418	2.0771	2.8759	3.3022	219.4646	6.3354	2.5150	3.4877	4.0066	320.2102	9.2437	3.6778	5.1138	5.8873
	0.5	-0.5	150.1921	4.3357	1.8260	2.6850	3.2045	181.6446	5.2436	2.2092	3.2503	3.8811	265.3795	7.6608	3.2266	4.7487	5.6820
		0	192.6358	5.5609	2.1535	2.9241	3.3141	232.8650	6.7222	2.6074	3.5450	4.0194	339.8836	9.8116	3.8123	5.1948	5.9030
		0.5	233.7441	6.7476	2.3993	3.0717	3.3827	282.4696	8.1542	2.9077	3.7276	4.1062	412.0243	11.8941	4.2578	5.4745	6.0418
III	-0.5	0	168.4502	4.8627	2.1888	3.5195	4.6090	203.8782	5.8855	2.6485	4.2576	5.5762	298.2031	8.6084	3.8694	6.2118	8.1375
		0.5	252.6705	7.2940	2.9581	4.2156	5.0168	305.2823	8.8127	3.5785	5.1044	6.0739	444.9805	12.8455	5.2230	7.4598	8.8797
	0	-0.5	187.9923	5.4269	2.3877	3.7206	4.7311	227.6383	6.5714	2.8896	4.4999	5.7229	333.2676	9.6206	4.2222	6.5621	8.3498
		0	274.3633	7.9202	3.1210	4.3382	5.0805	331.6381	9.5736	3.7760	5.2517	6.1503	483.8187	13.9666	5.5119	7.6723	8.9894
		0.5	355.3718	10.2587	3.6142	4.6668	5.2607	429.1580	12.3887	4.3752	5.6522	6.3696	624.9250	18.0400	6.3921	8.2670	9.3130
	0.5	-0.5	295.4843	8.5299	3.2666	4.4409	5.1332	357.3045	10.3145	3.9523	5.3751	6.2135	521.6564	15.0589	5.7691	7.8501	9.0800
		0	377.8274	10.9069	3.7280	4.7361	5.2934	456.4354	13.1762	4.5130	5.7351	6.4091	665.1059	19.2000	6.5929	8.3857	9.3698
		0.5	457.4936	13.2067	4.0524	4.9266	5.0300	552.3319	15.9444	4.9082	5.9665	6.1333	803.8422	23.2049	7.1775	8.7276	9.0753

* for general value of h_0

Table 4.7
Values of frequency parameter Ω for C-S plate for $\eta = -0.5$, $\varepsilon = 0.3$

Mode	α	β	μ = -0.5					μ = 0.0					μ = 1.0				
			*	Ωc	Ωs			*	Ωc	Ωs			*	Ωc	Ωs		
				h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2
I	-0.5	0	21.7966	0.6292	0.3109	0.6013	1.0723	25.4001	0.7332	0.3622	0.6997	1.2440	34.4067	0.9932	0.4901	0.9443	1.6652
		0.5	27.5965	0.7966	0.3905	0.7396	1.2471	32.0478	0.9251	0.4533	0.8573	1.4403	43.0805	1.2436	0.6085	1.1466	1.9080
	0	-0.5	24.6425	0.7114	0.3503	0.6714	1.1672	28.7521	0.8300	0.4086	0.7819	1.3543	39.0466	1.1272	0.5541	1.0567	1.8121
		0	30.6950	0.8861	0.4325	0.8100	1.3301	35.6845	1.0301	0.5024	0.9394	1.5361	48.0689	1.3876	0.6756	1.2572	2.0333
		0.5	35.8742	1.0356	0.5005	0.9153	1.4282	41.6072	1.2011	0.5800	1.0586	1.6444	55.7757	1.6101	0.7757	1.4081	2.1640
	0.5	-0.5	33.7394	0.9740	0.4731	0.8758	1.4017	39.2623	1.1334	0.5501	1.0162	1.6187	52.9898	1.5297	0.7407	1.3605	2.1411
		0	39.0813	1.1282	0.5421	0.9788	1.4906	45.3668	1.3096	0.6286	1.1323	1.7161	60.9152	1.7585	0.8417	1.5062	2.2564
		0.5	43.9772	1.2695	0.6029	1.0614	1.5459	50.9613	1.4711	0.6978	1.2252	1.7762	68.1932	1.9686	0.9306	1.6228	2.3281
II	-0.5	0	68.3245	1.9724	0.9578	1.7726	2.8472	80.2919	2.3178	1.1246	2.0771	3.3234	110.5832	3.1923	1.5455	2.8401	4.5031
		0.5	91.6160	2.6447	1.2591	2.2327	3.3249	107.3603	3.0992	1.4743	2.6097	3.8788	146.9902	4.2432	2.0137	3.5477	5.2444
	0	-0.5	77.0726	2.2249	1.0717	1.9470	3.0168	90.6331	2.6164	1.2588	2.2811	3.5204	125.0034	3.6085	1.7312	3.1176	4.7682
		0	101.0028	2.9157	1.3736	2.3859	3.4449	118.4502	3.4194	1.6091	2.7892	4.0203	162.4348	4.6891	2.1999	3.7919	5.4407
		0.5	122.4808	3.5357	1.6271	2.7135	3.7205	143.4003	4.1396	1.9032	3.1690	4.3445	195.9585	5.6568	2.5928	4.2971	5.8805
	0.5	-0.5	110.2397	3.1823	1.4825	2.5228	3.5427	129.3639	3.7344	1.7373	2.9491	4.1360	177.6378	5.1280	2.3765	4.0088	5.6032
		0	132.1634	3.8152	1.7330	2.8312	3.7930	154.8352	4.4697	2.0278	3.3069	4.4323	211.8704	6.1162	2.7643	4.4848	6.0085
		0.5	152.8064	4.4111	1.9519	3.0745	3.9725	178.8102	5.1618	2.2818	3.5899	4.6454	244.0686	7.0457	3.1034	4.8627	6.2984
III	-0.5	0	141.7322	4.0915	1.9399	3.4130	5.0371	166.9279	4.8188	2.2816	4.0035	5.8925	230.8120	6.6630	3.1437	5.4802	8.0158
		0.5	191.9412	5.5409	2.5343	4.1886	5.6918	225.2457	6.5023	2.9707	4.9015	6.6586	309.1649	8.9248	4.0638	6.6724	9.0503
	0	-0.5	160.0398	4.6200	2.1601	3.7029	5.2750	188.6265	5.4452	2.5413	4.3427	6.1729	261.1899	7.5399	3.5031	5.9414	8.4032
		0	211.6788	6.1106	2.7459	4.4207	5.8449	248.6033	7.1766	3.2200	5.1737	6.8415	341.7653	9.8659	4.4076	7.0439	9.3103
		0.5	257.8388	7.4432	3.2121	4.9098	6.1593	302.1835	8.7233	3.7607	5.7430	7.2132	413.6597	11.9413	5.1288	7.8058	9.8194
	0.5	-0.5	231.0865	6.6709	2.9431	4.6220	5.9705	271.5783	7.8398	3.4520	5.4098	6.9923	373.8542	10.7922	4.7267	7.3661	9.5263
		0	278.2564	8.0326	3.3952	5.0717	6.2423	326.3317	9.4204	3.9761	5.9335	7.3148	447.3232	12.9131	5.4250	8.0679	9.9693
		0.5	322.5429	9.3110	3.7708	5.4018	5.8220	377.7199	10.9038	4.4123	6.3199	7.0712	516.2298	14.9023	6.0074	8.5898	10.0247

* for general value of h_0

Table 4.8

Values of frequency parameter Ω for C-S plate for $\eta = 0.0$, $\varepsilon = 0.3$

Mode	α	β	μ = -0.5					μ = 0.0					μ = 1.0				
			*	Ωc	Ωs			*	Ωc	Ωs			*	Ωc	Ωs		
				h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2
I	-0.5	0	18.2248	0.5261	0.2600	0.5033	0.8996	21.2689	0.6140	0.3034	0.5866	1.0456	28.8937	0.8341	0.4118	0.7942	1.4049
		0.5	23.1568	0.6685	0.3278	0.6212	1.0491	26.9280	0.7773	0.3810	0.7211	1.2139	36.2929	1.0477	0.5128	0.9673	1.6139
	0	-0.5	20.6020	0.5947	0.2930	0.5621	0.9800	24.0735	0.6949	0.3422	0.6557	1.1393	32.7887	0.9465	0.4655	0.8891	1.5306
		0	25.7509	0.7434	0.3630	0.6804	1.1197	29.9777	0.8654	0.4223	0.7904	1.2956	40.4898	1.1688	0.5694	1.0611	1.7216
		0.5	30.1541	0.8705	0.4208	0.7700	1.2028	35.0181	1.0109	0.4883	0.8920	1.3875	47.0608	1.3585	0.6548	1.1902	1.8326
	0.5	-0.5	28.2983	0.8169	0.3970	0.7358	1.1807	32.9763	0.9519	0.4623	0.8551	1.3663	44.6278	1.2883	0.6243	1.1487	1.8144
		0	32.8413	0.9480	0.4557	0.8235	1.2562	38.1733	1.1020	0.5292	0.9543	1.4491	51.3880	1.4834	0.7105	1.2735	1.9125
		0.5	37.0030	1.0682	0.5074	0.8937	1.3020	42.9335	1.2394	0.5881	1.0333	1.4987	57.5915	1.6625	0.7865	1.3730	1.9713
II	-0.5	0	57.5716	1.6619	0.8076	1.4973	2.4140	67.7557	1.9559	0.9498	1.7578	2.8243	93.5959	2.7019	1.3095	2.4125	3.8444
		0.5	77.6408	2.2413	1.0676	1.8956	2.8289	91.1204	2.6304	1.2522	2.2203	3.3085	125.1332	3.6123	1.7162	3.0307	4.4955
	0	-0.5	64.8848	1.8731	0.9031	1.6443	2.5584	76.4146	2.2059	1.0625	1.9302	2.9926	105.7092	3.0516	1.4660	2.6485	4.0723
		0	85.5020	2.4682	1.1638	2.0249	2.9304	100.4228	2.8990	1.3656	2.3721	3.4285	138.1307	3.9875	1.8735	3.2386	4.6629
		0.5	104.0311	3.0031	1.3827	2.3085	3.1684	121.9854	3.5214	1.6203	2.7021	3.7097	167.2062	4.8268	2.2155	3.6806	5.0473
	0.5	-0.5	93.2348	2.6915	1.2552	2.1404	3.0130	109.5739	3.1631	1.4735	2.5076	3.5266	150.9197	4.3567	2.0228	3.4235	4.8014
		0	112.1456	3.2374	1.4717	2.4077	3.2290	131.5838	3.7985	1.7252	2.8188	3.7833	180.6065	5.2137	2.3606	3.8404	5.1556
		0.5	129.9655	3.7518	1.6609	2.6181	3.3830	152.3169	4.3970	1.9453	3.0645	3.9673	208.5503	6.0203	2.6561	4.1710	5.4076
III	-0.5	0	119.6727	3.4547	1.6398	2.8918	4.2806	141.1443	4.0745	1.9318	3.3989	5.0193	195.7110	5.6497	2.6704	4.6711	6.8592
		0.5	162.9444	4.7038	2.1530	3.5630	4.8445	191.4866	5.5277	2.5281	4.1785	5.6816	263.5701	7.6086	3.4704	5.7130	7.7605
	0	-0.5	135.0093	3.8974	1.8251	3.1374	4.4822	159.3503	4.6000	2.1507	3.6872	5.2574	221.2805	6.3878	2.9747	5.0655	7.1900
		0	179.5167	5.1822	2.3313	3.7592	4.9726	211.1294	6.0948	2.7387	4.4095	5.8351	291.0727	8.4025	3.7622	6.0302	7.9800
		0.5	219.3431	6.3319	2.7338	4.1810	5.2366	257.4324	7.4314	3.2067	4.9022	6.1494	353.4025	10.2019	4.3901	6.6945	8.4157
	0.5	-0.5	195.8031	5.6523	2.4974	3.9296	5.0775	230.4400	6.6522	2.9347	4.6100	5.9613	318.1296	9.1836	4.0333	6.3055	8.1619
		0	236.5002	6.8272	2.8880	4.3174	5.3020	277.7565	8.0181	3.3888	5.0634	6.2312	381.8257	11.0224	4.6418	6.9176	8.5392
		0.5	274.7322	7.9308	3.2119	4.6006	4.7351	322.1903	9.3008	3.7662	5.3962	5.7927	441.5969	12.7478	5.1489	7.3708	8.4441

* for general value of h_0

Table 4.9

Values of frequency parameter Ω for C-S plate for $\eta = 1.0$, $\varepsilon = 0.3$

Mode	α	β	μ = -0.5					μ = 0.0					μ = 1.0				
			*	Ωc	Ωs			*	Ωc	Ωs			*	Ωc	Ωs		
				h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2
I	-0.5	0	12.6709	0.3658	0.1809	0.3505	0.6289	14.8296	0.4281	0.2116	0.4098	0.7337	20.2600	0.5849	0.2889	0.5583	0.9930
		0.5	16.2132	0.4680	0.2296	0.4354	0.7369	18.9029	0.5457	0.2676	0.5070	0.8558	25.6078	0.7392	0.3621	0.6841	1.1459
	0	-0.5	14.3202	0.4134	0.2038	0.3916	0.6860	16.7817	0.4844	0.2387	0.4583	0.8007	22.9887	0.6636	0.3267	0.6254	1.0840
		0	18.0210	0.5202	0.2541	0.4770	0.7875	21.0352	0.6072	0.2965	0.5558	0.9147	28.5611	0.8245	0.4020	0.7509	1.2245
		0.5	21.1826	0.6115	0.2956	0.5413	0.8461	24.6614	0.7119	0.3440	0.6290	0.9797	33.3061	0.9615	0.4638	0.8444	1.3032
	0.5	-0.5	19.7943	0.5714	0.2779	0.5160	0.8313	23.1294	0.6677	0.3245	0.6016	0.9659	31.4699	0.9085	0.4407	0.8134	1.2927
		0	23.0582	0.6656	0.3201	0.5791	0.8847	26.8709	0.7757	0.3728	0.6732	1.0245	36.3557	1.0495	0.5032	0.9041	1.3621
		0.5	26.0457	0.7519	0.3571	0.6291	0.9153	30.2945	0.8745	0.4151	0.7297	1.0574	40.8329	1.1787	0.5580	0.9756	1.4004
II	-0.5	0	40.7035	1.1750	0.5716	1.0627	1.7236	48.0453	1.3869	0.6744	1.2522	2.0259	66.7647	1.9273	0.9358	1.7314	2.7833
		0.5	55.5314	1.6031	0.7641	1.3588	2.0333	65.3699	1.8871	0.8992	1.5981	2.3901	90.3191	2.6073	1.2409	2.1995	3.2805
	0	-0.5	45.7879	1.3218	0.6383	1.1663	1.8271	54.0843	1.5613	0.7534	1.3744	2.1473	75.2669	2.1728	1.0463	1.9005	2.9498
		0	61.0156	1.7614	0.8313	1.4498	2.1048	71.8801	2.0750	0.9789	1.7056	2.4753	99.4741	2.8716	1.3523	2.3485	3.4008
		0.5	74.7417	2.1576	0.9937	1.6604	2.2803	87.9111	2.5378	1.1687	1.9524	2.6845	121.2491	3.5002	1.6098	2.6838	3.6916
	0.5	-0.5	66.4065	1.9170	0.8954	1.5313	2.1629	78.2798	2.2597	1.0548	1.8018	2.5447	108.4751	3.1314	1.4583	2.4815	3.5001
		0	80.4101	2.3212	1.0560	1.7301	2.3219	94.6373	2.7319	1.2425	2.0349	2.7355	130.7003	3.7730	1.7128	2.7984	3.7679
		0.5	93.6291	2.7028	1.1961	1.8858	2.4339	110.0732	3.1775	1.4065	2.2183	2.8711	151.6573	4.3780	1.9356	3.0489	3.9569
III	-0.5	0	84.9784	2.4531	1.1664	2.0643	3.0700	100.5056	2.9013	1.3785	2.4358	3.6168	140.1478	4.0457	1.9179	3.3743	4.9890
		0.5	116.9681	3.3766	1.5464	2.5620	3.4834	137.8453	3.9793	1.8221	3.0179	4.1062	190.8137	5.5083	2.5187	4.1625	5.6655
	0	-0.5	95.6908	2.7624	1.2966	2.2388	3.2132	113.2620	3.2696	1.5331	2.6420	3.7869	158.1770	4.5662	2.1347	3.6598	5.2278
		0	128.5956	3.7122	1.6719	2.7008	3.5722	151.6705	4.3784	1.9711	3.1824	4.2132	210.2940	6.0707	2.7272	4.3915	5.8202
		0.5	158.1092	4.5642	1.9695	3.0109	3.7362	186.0954	5.3721	2.3192	3.5474	4.4255	256.9369	7.4171	3.1994	4.8911	6.1305
	0.5	-0.5	140.0103	4.0417	1.7890	2.8215	3.6444	165.2473	4.7703	2.1099	3.3253	4.3006	229.4380	6.6233	2.9213	4.5904	5.9479
		0	170.1657	4.9123	2.0781	3.1068	5.3253	200.4227	5.7857	2.4480	3.6614	4.4642	277.1019	7.9992	3.3797	5.0511	6.2095
		0.5	198.5331	5.7312	2.3168	3.3127	3.0994	233.4984	6.7405	2.7280	3.9055	3.8200	321.8825	9.2919	3.7607	5.3889	5.7573

* for general value of h_0

Table 4.10
Values of frequency parameter Ω for C-S plate for $\eta = -0.5$, $\varepsilon = 0.5$

Mode	α	β	μ = -0.5					μ = 0.0					μ = 1.0				
			*	Ωc	Ωs			*	Ωc	Ωs			*	Ωc	Ωs		
				h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2
I	-0.5	0	39.6941	1.1459	0.5625	1.0691	1.8187	47.5364	1.3723	0.6734	1.2787	2.1700	68.0839	1.9654	0.9637	1.8256	3.0784
		0.5	54.9213	1.5854	0.7644	1.3908	2.1484	65.6351	1.8947	0.9130	1.6587	2.5539	93.5726	2.7012	1.2996	2.3522	3.5923
	0	-0.5	45.1292	1.3028	0.6362	1.1935	1.9664	54.0850	1.5613	0.7622	1.4280	2.3461	77.5879	2.2398	1.0921	2.0401	3.3271
		0	60.8462	1.7565	0.8407	1.5052	2.2583	72.7697	2.1007	1.0047	1.7957	2.6847	103.9049	2.9995	1.4318	2.5476	3.7758
		0.5	75.3621	2.1755	1.0165	1.7339	2.4085	90.0093	2.5983	1.2127	2.0641	2.8557	128.1524	3.6994	1.7218	2.9136	3.9942
	0.5	-0.5	66.6429	1.9238	0.9137	1.6094	2.3503	79.7525	2.3023	1.0924	1.9204	2.7942	114.0269	3.2917	1.5582	2.7250	3.9296
		0	81.4569	2.3515	1.0883	1.8251	2.4809	97.3475	2.8102	1.2990	2.1729	2.9421	138.7734	4.0060	1.8456	3.0675	4.1156
		0.5	95.6754	2.7619	1.2417	1.9855	2.5443	114.2292	3.2975	1.4801	2.3596	3.0110	162.5071	4.6912	2.0968	3.3179	4.1956
II	-0.5	0	125.0411	3.6096	1.7225	3.0676	4.5914	150.6337	4.3484	2.0733	3.6860	5.5035	218.2967	6.3017	2.9986	5.3085	7.8823
		0.5	181.7712	5.2473	2.3958	3.9434	5.3165	218.6153	6.3109	2.8787	4.7313	6.3767	315.7372	9.1145	4.1473	6.7903	9.1398
	0	-0.5	140.9040	4.0675	1.9170	3.3301	4.7981	169.8328	4.9027	2.3079	4.0003	5.7500	246.3880	7.1126	3.3392	5.7576	8.2330
		0	198.9461	5.7431	2.5785	4.1377	5.4300	239.3897	6.9106	3.0988	4.9640	6.5149	346.0964	9.9909	4.4656	7.1220	9.3451
		0.5	253.5987	7.3208	3.1153	4.6768	5.7820	304.8783	8.8011	3.7406	5.6093	6.9497	439.9515	12.7003	5.3790	8.0410	9.9923
	0.5	-0.5	215.7791	6.2290	2.7485	4.3053	5.5196	259.7536	7.4984	3.3033	5.1642	6.6247	375.8649	10.8503	4.7605	7.4070	9.5109
		0	271.2516	7.8304	3.2661	4.7993	5.8302	326.2259	9.4173	3.9219	5.7562	7.0140	471.1341	13.6005	5.6398	8.2515	10.1017
		0.5	325.0357	9.3830	3.6924	5.1513	5.9154	390.6724	11.2777	4.4316	6.1800	7.2243	563.4914	16.2666	6.3646	8.8613	10.4932
III	-0.5	0	259.3575	7.4870	3.4395	5.7220	7.8414	312.9243	9.0333	4.1445	6.8806	9.4198	454.7849	13.1285	6.0043	9.9198	13.5496
		0.5	380.6252	10.9877	4.6790	7.0419	8.7226	458.2599	13.2288	5.6267	8.4593	10.4866	663.1366	19.1431	8.1155	12.1661	15.1014
	0	-0.5	291.8464	8.4249	3.7904	6.1095	8.1039	352.3468	10.1714	4.5680	7.3454	9.7379	512.7314	14.8013	6.6192	10.5856	14.0165
		0	416.0283	12.0097	4.9808	7.2946	8.8468	501.1745	14.4677	5.9904	8.7631	10.6398	726.0872	20.9603	8.6413	12.6042	15.3335
		0.5	532.7079	15.3780	5.8635	7.9996	7.5308	640.9961	18.5040	7.0464	9.6119	9.2796	926.4917	26.7455	10.1454	13.8269	14.0129
	0.5	-0.5	450.6690	13.0097	5.2535	7.5081	8.9339	543.1765	15.6802	6.3186	9.0202	10.7505	787.7330	22.7399	9.1146	12.9758	15.5067
		0	569.1777	16.4307	6.0868	8.1448	7.3741	685.1907	19.7798	7.3150	9.7877	9.0573	991.2811	28.6158	10.5323	14.0840	13.6066
		0.5	683.9349	19.7435	6.7225	8.5536	6.2284	822.7019	23.7494	8.0768	10.2827	7.5597	1188.3558	34.3049	11.6198	14.8044	11.3618

* for general value of h_0

Table 4.11

Values of frequency parameter Ω for C-S plate for $\eta = 0.0$, $\varepsilon = 0.5$

Mode	α	β	$\mu = -0.5$					$\mu = 0.0$					$\mu = 1.0$				
			*	Ωc	Ωs			*	Ωc	Ωs			*	Ωc	Ωs		
				h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2
I	-0.5	0	32.5465	0.9395	0.4614	0.8775	1.4956	39.0056	1.1260	0.5527	1.0504	1.7863	55.9483	1.6151	0.7922	1.5022	2.5394
		0.5	45.1359	1.3030	0.6284	1.1441	1.7698	53.9781	1.5582	0.7511	1.3657	2.1060	77.0590	2.2245	1.0707	1.9400	2.9684
	0	-0.5	36.9859	1.0677	0.5216	0.9794	1.6178	44.3594	1.2805	0.6254	1.1730	1.9323	63.7318	1.8398	0.8975	1.6787	2.7463
		0	49.9829	1.4429	0.6909	1.2382	1.8611	59.8199	1.7269	0.8263	1.4786	2.2149	85.5340	2.4692	1.1794	2.1014	3.1218
		0.5	61.9818	1.7893	0.8363	1.4278	1.9847	74.0786	2.1385	0.9986	1.7012	2.3557	105.6116	3.0487	1.4199	2.4058	3.3014
	0.5	-0.5	54.7231	1.5797	0.7507	1.3240	1.9376	65.5351	1.8918	0.8983	1.5813	2.3063	93.8331	2.7087	1.2833	2.2481	3.2506
		0	66.9700	1.9333	0.8952	1.5030	2.0453	80.0898	2.3120	1.0694	1.7911	2.4280	114.3272	3.3003	1.5219	2.5334	3.4035
		0.5	78.7220	2.2725	1.0222	1.6357	2.0954	94.0511	2.7150	1.2195	1.9458	2.4822	133.9771	3.8676	1.7302	2.7412	3.4651
II	-0.5	0	103.1078	2.9765	1.4214	2.5354	3.8051	124.3055	3.5884	1.7123	3.0496	4.5668	180.4168	5.2082	2.4808	4.4012	6.5571
		0.5	150.4019	4.3417	1.9837	3.2694	4.4132	181.0269	5.2258	2.3859	3.9273	5.3006	261.8540	7.5591	3.4437	5.6496	7.6179
	0	-0.5	116.0819	3.3510	1.5809	2.7519	3.9766	140.0209	4.0421	1.9050	3.3094	4.7718	203.4483	5.8730	2.7610	4.7735	6.8498
		0	164.4670	4.7478	2.1338	3.4298	4.5066	198.0535	5.7173	2.5669	4.1197	5.4144	286.7766	8.2785	3.7061	5.9250	7.7875
		0.5	210.0397	6.0633	2.5825	3.8815	4.7987	252.7074	7.2950	3.1041	4.6615	5.7768	365.2350	10.5434	4.4731	6.6995	8.3297
	0.5	-0.5	178.2476	5.1456	2.2735	3.5684	4.5801	214.7381	6.1990	2.7351	4.2856	5.5047	311.2060	8.9837	3.9495	6.1620	7.9241
		0	224.5017	6.4808	2.7066	3.9827	4.8364	270.2104	7.8003	3.2535	4.7830	5.8280	390.8431	11.2827	4.6886	6.8742	8.4187
		0.5	269.3548	7.7756	3.0630	4.2770	4.8248	324.0000	9.3531	3.6804	5.1382	5.9479	468.0570	13.5116	5.2978	7.3873	8.7453
III	-0.5	0	214.1751	6.1827	2.8436	4.7396	6.5049	258.5950	7.4650	3.4294	5.7056	7.8239	376.3659	10.8647	4.9771	8.2440	11.2815
		0.5	315.3313	9.1028	3.8797	5.8447	7.2345	379.9217	10.9674	4.6702	7.0298	8.7092	550.5688	15.8936	6.7493	10.1349	12.5751
	0	-0.5	240.7904	6.9510	3.1323	5.0605	6.7212	290.9158	8.3980	3.7783	6.0910	8.0863	423.9489	12.2384	5.4850	8.7980	11.6673
		0	344.3759	9.9413	4.1284	6.0534	7.3325	415.1567	11.9845	4.9703	7.2811	8.8312	602.3365	17.3880	7.1845	10.4983	12.7620
		0.5	441.7266	12.7515	4.8662	6.6407	6.0690	531.9052	15.3548	5.8545	7.9896	7.4950	769.9272	22.2259	8.4480	11.5228	11.3847
	0.5	-0.5	372.7839	10.7613	4.3533	6.2298	7.3927	449.6286	12.9797	5.2414	7.4937	8.9129	653.0109	18.8508	7.5768	10.8065	12.8967
		0	471.6584	13.6156	5.0504	6.7601	5.9565	568.2058	16.4027	6.0764	8.1342	7.3316	823.2291	23.7646	8.7688	11.7348	11.0742
		0.5	567.4145	16.3798	5.5812	7.0973	5.1133	683.0359	19.7175	6.7137	8.5435	6.1683	988.0491	28.5225	9.6820	12.3330	9.2218

* for general value of h_0

Table 4.12

Values of frequency parameter Ω for C-S plate for $\eta = 1.0$, $\varepsilon = 0.5$

Mode	α	β	μ = -0.5					μ = 0.0					μ = 1.0				
			*	Ωc	Ωs			*	Ωc	Ωs			*	Ωc	Ωs		
				h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2
I	-0.5	0	21.8180	0.6298	0.3094	0.5892	1.0077	26.1864	0.7559	0.3713	0.7065	1.2060	37.6703	1.0874	0.5337	1.0138	1.7215
		0.5	30.3957	0.8774	0.4234	0.7716	1.1959	36.4000	1.0508	0.5068	0.9226	1.4260	52.1048	1.5041	0.7245	1.3150	2.0181
	0	-0.5	24.7715	0.7151	0.3496	0.6576	1.0908	29.7544	0.8589	0.4198	0.7889	1.3056	42.8756	1.2377	0.6044	1.1330	1.8637
		0	33.6303	0.9708	0.4652	0.8350	1.2585	40.3052	1.1635	0.5572	0.9988	1.5010	57.7899	1.6683	0.7977	1.4246	2.1246
		0.5	41.8027	1.2067	0.5643	0.9644	1.3413	50.0281	1.4442	0.6749	1.1512	1.5953	71.5116	2.0644	0.9625	1.6339	2.2445
	0.5	-0.5	36.7910	1.0621	0.5052	0.8928	1.3112	44.1226	1.2737	0.6054	1.0683	1.5642	63.3525	1.8288	0.8677	1.5245	2.2145
		0	45.1345	1.3029	0.6038	1.0153	1.3833	54.0501	1.5603	0.7225	1.2123	1.6457	77.3633	2.2333	1.0313	1.7212	2.3162
		0.5	53.1374	1.5339	0.6904	1.1056	1.4138	63.5684	1.8351	0.8249	1.3177	1.6778	90.7897	2.6209	1.1741	1.8633	2.3507
II	-0.5	0	69.9567	2.0195	0.9655	1.7268	2.6036	84.4669	2.4383	1.1651	2.0814	3.1328	122.9695	3.5498	1.6938	3.0164	4.5211
		0.5	102.7516	2.9662	1.3565	2.2399	3.0289	123.8654	3.5757	1.6346	2.6970	3.6481	179.7277	5.1883	2.3682	3.8982	5.2718
	0	-0.5	78.6130	2.2694	1.0724	1.8734	2.7211	94.9689	2.7415	1.2946	2.2578	3.2737	138.4099	3.9955	1.8829	3.2709	4.7235
		0	112.1592	3.2378	1.4574	2.3486	3.0916	135.2719	3.9050	1.7565	2.8278	3.7248	196.4776	5.6718	2.5459	4.0868	5.3867
		0.5	143.7771	4.1505	1.7696	2.6639	3.2904	173.2542	5.0014	2.1315	3.2076	3.9745	251.1881	7.2512	3.0844	4.6340	5.7655
	0.5	-0.5	121.3705	3.5037	1.5513	2.4427	3.1406	146.4420	4.2274	1.8700	2.9409	3.7852	212.8851	6.1455	2.7110	4.2495	5.4788
		0	153.4572	4.4299	1.8531	2.7323	3.3116	184.9883	5.3402	2.2323	3.2901	4.0056	268.4112	7.7484	3.2308	4.7535	5.8235
		0.5	184.5831	5.3285	2.1012	2.9370	3.1520	222.3788	6.4195	2.5305	3.5382	3.9148	322.2663	9.3030	3.6593	5.1147	9.5501
III	-0.5	0	145.7544	4.2076	1.9387	3.2414	4.4598	176.2358	5.0875	2.3422	3.9107	5.3776	257.2359	7.4258	3.4112	5.6758	7.7926
		0.5	215.9868	6.2350	2.6600	4.0118	4.9528	260.6043	7.5230	3.2083	4.8373	5.9803	378.7528	10.9337	4.6551	7.0085	8.6835
	0	-0.5	163.5737	4.7220	2.1334	3.4602	4.6059	197.9086	5.7131	2.5781	4.1744	5.5552	289.2433	8.3497	3.7564	6.0572	8.0549
		0	235.4879	6.7980	2.8281	4.1534	5.0040	284.2988	8.2070	3.4117	5.0084	6.0514	413.6757	11.9418	4.9519	7.2574	8.8006
		0.5	303.1119	8.7501	3.3410	4.5587	3.9205	365.5229	10.5517	4.0285	5.4992	4.8578	530.6324	15.3180	5.8393	7.9726	7.4475
	0.5	-0.5	254.5464	7.3481	2.9803	4.2731	4.9882	307.4620	8.8757	3.5958	5.1531	6.0705	447.8340	12.9279	5.2201	7.4683	8.8710
		0	323.2242	9.3307	3.4656	4.6388	3.8655	389.9532	11.2570	4.1792	5.5966	4.7727	566.6192	16.3569	6.0586	8.1161	7.2722
		0.5	389.7552	11.2513	3.8336	4.8652	3.4794	469.8596	13.5637	4.6227	5.8730	4.1918	681.6648	19.6780	6.6990	8.5245	11.2516

* for general value of h_0

Table 4.13

Values of frequency parameter Ω for C-F plate for $\eta = -0.5$, $\varepsilon = 0.3$

Mode	α	β	μ = -0.5 *	μ = -0.5 Ωc h_0=0.1	μ = -0.5 Ωs h_0=0.05	μ = -0.5 Ωs h_0=0.1	μ = -0.5 Ωs h_0=0.2	μ = 0.0 *	μ = 0.0 Ωc h_0=0.1	μ = 0.0 Ωs h_0=0.05	μ = 0.0 Ωs h_0=0.1	μ = 0.0 Ωs h_0=0.2	μ = 1.0 *	μ = 1.0 Ωc h_0=0.1	μ = 1.0 Ωs h_0=0.05	μ = 1.0 Ωs h_0=0.1	μ = 1.0 Ωs h_0=0.2
I	-0.5	0	6.3679	0.1838	0.0916	0.1814	0.3493	7.1862	0.2074	0.1034	0.2047	0.3944	9.1377	0.2638	0.1315	0.2604	0.5017
		0.5	6.3069	0.1821	0.0906	0.1788	0.3403	7.1384	0.2061	0.1026	0.2024	0.3852	9.1759	0.2649	0.1318	0.2601	0.4947
	0	-0.5	7.5284	0.2173	0.1082	0.2135	0.4062	8.4824	0.2449	0.1219	0.2406	0.4580	10.7501	0.3103	0.1545	0.3049	0.5807
		0	7.3211	0.2113	0.1051	0.2066	0.3884	8.2692	0.2387	0.1187	0.2334	0.4389	10.5766	0.3053	0.1518	0.2985	0.5610
		0.5	7.5190	0.2171	0.1077	0.2109	0.3907	8.5316	0.2463	0.1222	0.2393	0.4431	11.0521	0.3190	0.1583	0.3097	0.5721
	0.5	-0.5	8.3602	0.2413	0.1198	0.2347	0.4354	9.4286	0.2722	0.1351	0.2647	0.4914	12.0150	0.3468	0.1722	0.3373	0.6259
		0	8.4694	0.2445	0.1212	0.2362	0.4319	9.5890	0.2768	0.1372	0.2674	0.4889	12.3565	0.3567	0.1767	0.3443	0.6282
		0.5	8.7765	0.2534	0.1253	0.2429	0.4369	9.9821	0.2882	0.1425	0.2762	0.4963	13.0148	0.3757	0.1857	0.3594	0.6429
II	-0.5	0	32.1621	0.9284	0.4570	0.8748	1.5162	37.3807	1.0791	0.5309	1.0147	1.7512	50.3031	1.4521	0.7134	1.3585	2.3206
		0.5	38.9445	1.1242	0.5477	1.0207	1.6552	45.0632	1.3009	0.6333	1.1784	1.9031	60.1385	1.7360	0.8438	1.5635	2.5007
	0	-0.5	37.3159	1.0772	0.5278	0.9981	1.6774	43.4068	1.2530	0.6135	1.1578	1.9355	58.4901	1.6885	0.8251	1.5493	2.5574
		0	44.0029	1.2703	0.6152	1.1307	1.7803	50.9530	1.4709	0.7117	1.3053	2.0451	68.0742	1.9651	0.9487	1.7305	2.6804
		0.5	50.3005	1.4521	0.6940	1.2377	1.8332	58.1096	1.6775	0.8010	1.4253	2.1020	77.3162	2.2319	1.0631	1.8813	2.7486
	0.5	-0.5	49.0950	1.4173	0.6820	1.2353	1.8910	56.8871	1.6422	0.7893	1.4258	2.1703	76.0830	2.1963	1.0526	1.8886	2.8379
		0	55.3906	1.5990	0.7586	1.3321	1.9238	64.0251	1.8482	0.8757	1.5335	2.2039	85.2544	2.4611	1.1624	2.0217	2.8753
		0.5	61.4448	1.7738	0.8283	1.4091	1.9269	70.9165	2.0472	0.9548	1.6199	2.2064	94.1944	2.7192	1.2640	2.1304	2.8805
III	-0.5	0	85.6673	2.4730	1.1948	2.1864	3.4430	100.5458	2.9025	1.4009	2.5575	4.0112	138.0319	3.9846	1.9183	3.4820	5.4110
		0.5	112.0779	3.2354	1.5273	2.6643	3.8719	131.0039	3.7818	1.7835	3.1052	4.5025	178.3613	5.1488	2.4218	4.1947	6.0486
	0	-0.5	97.9669	2.8281	1.3534	2.4264	3.6903	115.0964	3.3225	1.5879	2.8384	4.2996	158.3151	4.5702	2.1767	3.8638	5.8005
		0	124.7385	3.6009	1.6795	2.8664	4.0498	145.9411	4.2130	1.9625	3.3413	4.7108	199.0713	5.7467	2.6676	4.5140	6.3320
		0.5	149.2424	4.3083	1.9552	3.1906	4.2480	174.2115	5.0291	2.2798	3.7135	4.9411	236.5409	6.8283	3.0851	4.9998	6.6378
	0.5	-0.5	137.3316	3.9644	1.8256	3.0497	4.2048	160.8073	4.6421	2.1341	3.5551	4.8928	219.7064	6.3424	2.9030	4.8025	6.5811
		0	162.1695	4.6814	2.0937	3.3461	4.3686	189.4485	5.4689	2.4423	3.8949	5.0831	257.6261	7.4370	3.3071	5.2445	6.8326
		0.5	185.7701	5.3627	2.3267	3.5723	4.4254	216.6802	6.2550	2.7105	4.1553	5.1553	293.7314	8.4793	3.6600	5.5859	6.9397

* for general value of h_0

Table 4.14

Values of frequency parameter Ω for C-F plate for $\eta = 0.0$, $\varepsilon = 0.3$

Mode	α	β	μ = -0.5					μ = 0.0					μ = 1.0				
			*	Ωc	Ωs			*	Ωc	Ωs			*	Ωc	Ωs		
				h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2
I	-0.5	0	5.1283	0.1480	0.0738	0.1461	0.2818	5.7908	0.1672	0.0833	0.1651	0.3184	7.3717	0.2128	0.1061	0.2101	0.4054
		0.5	5.0729	0.1464	0.0729	0.1439	0.2742	5.7443	0.1658	0.0825	0.1630	0.3106	7.3904	0.2133	0.1062	0.2096	0.3992
	0	-0.5	6.0715	0.1753	0.0873	0.1723	0.3283	6.8452	0.1976	0.0984	0.1943	0.3704	8.6853	0.2507	0.1248	0.2465	0.4703
		0	5.8939	0.1701	0.0846	0.1664	0.3134	6.6604	0.1923	0.0956	0.1881	0.3543	8.5265	0.2461	0.1224	0.2408	0.4534
		0.5	6.0497	0.1746	0.0867	0.1698	0.3151	6.8672	0.1982	0.0984	0.1927	0.3576	8.9031	0.2570	0.1276	0.2497	0.4622
	0.5	-0.5	6.7360	0.1945	0.0965	0.1892	0.3518	7.6004	0.2194	0.1089	0.2136	0.3972	9.6943	0.2799	0.1390	0.2724	0.5066
		0	6.8183	0.1968	0.0976	0.1903	0.3487	7.7229	0.2229	0.1105	0.2156	0.3950	9.9600	0.2875	0.1425	0.2778	0.5081
		0.5	7.0629	0.2039	0.1009	0.1957	0.3527	8.0361	0.2320	0.1148	0.2226	0.4009	10.4858	0.3027	0.1497	0.2899	0.5199
II	-0.5	0	26.7633	0.7726	0.3805	0.7291	1.2677	31.1470	0.8991	0.4426	0.8470	1.4670	42.0253	1.2132	0.5964	1.1374	1.9512
		0.5	32.5501	0.9396	0.4579	0.8541	1.3871	37.7109	1.0886	0.5302	0.9875	1.5978	50.4485	1.4563	0.7082	1.3140	2.1070
	0	-0.5	31.0322	0.8958	0.4392	0.8319	1.4037	36.1468	1.0435	0.5112	0.9665	1.6229	48.8414	1.4099	0.6895	1.2976	2.1531
		0	36.7545	1.0610	0.5141	0.9461	1.4930	42.6142	1.2302	0.5956	1.0939	1.7183	57.0765	1.6477	0.7961	1.4548	2.2607
		0.5	42.1147	1.2157	0.5812	1.0371	1.5360	48.7119	1.4062	0.6717	1.1962	1.7645	64.9641	1.8754	0.8938	1.5837	2.3154
	0.5	-0.5	40.9838	1.1831	0.5697	1.0337	1.5869	47.5510	1.3727	0.6604	1.1951	1.8250	63.7613	1.8406	0.8831	1.5882	2.3957
		0	46.3513	1.3380	0.6351	1.1163	1.6131	53.6434	1.5486	0.7342	1.2873	1.8515	71.6032	2.0670	0.9772	1.7025	2.4242
		0.5	51.4989	1.4866	0.6944	1.1813	1.6119	59.5079	1.7178	0.8015	1.3602	1.8491	79.2214	2.2869	1.0638	1.7945	2.4222
III	-0.5	0	72.0221	2.0791	1.0054	1.8435	2.9141	84.6509	2.4437	1.1806	2.1604	3.4027	116.5453	3.3644	1.6218	2.9524	4.6108
		0.5	94.7960	2.7365	1.2927	2.2581	3.2865	110.9653	3.2033	1.5120	2.6371	3.8312	151.5210	4.3740	2.0600	3.5769	5.1713
	0	-0.5	82.2690	2.3749	1.1379	2.0454	3.1236	96.7923	2.7942	1.3372	2.3973	3.6477	133.5253	3.8545	1.8391	3.2762	4.9434
		0	105.3744	3.0419	1.4202	2.4283	3.4365	123.4661	3.5642	1.6623	2.8365	4.0073	168.9110	4.8760	2.2673	3.8483	5.4124
		0.5	126.5156	3.6522	1.6584	2.7089	3.6036	147.9017	4.2696	1.9372	3.1600	4.2028	201.4156	5.8144	2.6310	4.2735	5.6750
	0.5	-0.5	115.8894	3.3454	1.5426	2.5828	3.5672	135.8991	3.9231	1.8065	3.0174	4.1611	186.2261	5.3759	2.4660	4.0939	5.6240
		0	137.3273	3.9643	1.7746	2.8400	3.7045	160.6664	4.6380	2.0739	3.3133	4.3220	219.1391	6.3260	2.8187	4.4817	5.8396
		0.5	157.6965	4.5523	1.9758	3.0349	3.7424	184.2118	5.3177	2.3063	3.5387	4.3735	250.4677	7.2304	3.1262	4.7794	5.9220

* for general value of h_0

Table 4.15

Values of frequency parameter Ω for C-F plate for $\eta = 1.0$, $\varepsilon = 0.3$

Mode	α	β	μ = -0.5					μ = 0.0					μ = 1.0				
			*	Ωc	Ωs			*	Ωc	Ωs			*	Ωc	Ωs		
				h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2
I	-0.5	0	3.3114	0.0956	0.0476	0.0944	0.1824	3.7434	0.1081	0.0539	0.1068	0.2064	4.7755	0.1379	0.0687	0.1362	0.2634
		0.5	3.2683	0.0943	0.0470	0.0928	0.1772	3.7040	0.1069	0.0532	0.1052	0.2009	4.7730	0.1378	0.0686	0.1355	0.2587
	0	-0.5	3.9310	0.1135	0.0565	0.1116	0.2133	4.4370	0.1281	0.0638	0.1261	0.2410	5.6419	0.1629	0.0811	0.1603	0.3067
		0	3.8036	0.1098	0.0546	0.1075	0.2030	4.3019	0.1242	0.0618	0.1216	0.2298	5.5162	0.1592	0.0792	0.1560	0.2946
		0.5	3.8998	0.1126	0.0559	0.1096	0.2040	4.4301	0.1279	0.0635	0.1245	0.2317	5.7518	0.1660	0.0824	0.1615	0.3001
	0.5	-0.5	4.3536	0.1257	0.0624	0.1225	0.2284	4.9167	0.1419	0.0705	0.1383	0.2582	6.2817	0.1813	0.0901	0.1768	0.3300
		0	4.4001	0.1270	0.0630	0.1230	0.2262	4.9876	0.1440	0.0714	0.1395	0.2565	6.4419	0.1860	0.0922	0.1800	0.3306
		0.5	4.5546	0.1315	0.0651	0.1264	0.2287	5.1859	0.1497	0.0741	0.1439	0.2602	6.7761	0.1956	0.0968	0.1877	0.3382
II	-0.5	0	50.7219	1.4642	0.2626	0.5041	0.8810	59.7858	1.7259	0.3062	0.5873	1.0232	29.2102	0.8432	0.4149	0.7935	1.3711
		0.5	67.5763	1.9508	0.3186	0.5948	0.9673	26.2958	0.7591	0.3699	0.6898	1.1184	35.3481	1.0204	0.4966	0.9232	1.4853
	0	-0.5	57.8037	1.6686	0.3028	0.5750	0.9768	24.9634	0.7206	0.3535	0.6702	1.1339	33.9147	0.9790	0.4795	0.9057	1.5163
		0	25.5343	0.7371	0.3574	0.6588	1.0425	29.6805	0.8568	0.4152	0.7641	1.2045	39.9526	1.1533	0.5580	1.0225	1.5965
		0.5	29.3946	0.8485	0.4057	0.7239	1.0697	34.0810	0.9838	0.4701	0.8376	1.2334	45.6647	1.3182	0.6288	1.1156	1.6298
	0.5	-0.5	28.4399	0.8210	0.3958	0.7198	1.1095	33.0832	0.9550	0.4600	0.8350	1.2811	44.5905	1.2872	0.6187	1.1169	1.6948
		0	32.3175	0.9329	0.4430	0.7793	1.1248	37.4943	1.0824	0.5135	0.9016	1.2960	50.2893	1.4517	0.6872	1.2000	1.7092
		0.5	36.0177	1.0397	0.4854	0.8248	1.1179	41.7175	1.2043	0.5618	0.9528	1.2870	55.7911	1.6106	0.7496	1.2647	1.6973
III	-0.5	0	99.0326	2.8588	0.7090	1.3043	2.0745	117.0318	3.3784	0.8352	1.5341	2.4334	82.7881	2.3899	1.1545	2.1122	3.3275
		0.5	134.5223	3.8833	0.9222	1.6135	2.3518	79.3358	2.2902	1.0823	1.8920	2.7553	108.9750	3.1458	1.4845	2.5874	3.7557
	0	-0.5	112.3135	3.2422	0.8011	1.4459	2.2234	68.2032	1.9689	0.9443	1.7013	2.6086	94.6360	2.7319	1.3073	2.3433	3.5680
		0	74.9283	2.1630	1.0111	1.7331	2.4574	88.0520	2.5418	1.1875	2.0330	2.8800	121.1807	3.4982	1.6310	2.7817	3.9285
		0.5	90.5972	2.6153	1.1878	1.9412	2.5731	106.2304	3.0666	1.3926	2.2747	3.0177	145.5423	4.2014	1.9051	3.1042	4.1189
	0.5	-0.5	82.2269	2.3737	1.0965	1.8419	2.5494	96.7095	2.7918	1.2886	2.1612	2.9889	133.3173	3.8485	1.7716	2.9579	4.0800
		0	98.1254	2.8326	1.2689	2.0333	2.6427	115.1474	3.3240	1.4885	2.3832	3.1006	158.0067	4.5613	2.0383	3.2534	4.2350
		0.5	113.2356	3.2688	1.4180	2.1764	2.6505	132.6782	3.8301	1.6616	2.5501	3.1192	181.5034	5.2396	2.2701	3.4775	4.2779

* for general value of h_0

Table 4.16

Values of frequency parameter Ω for C-F plate for $\eta = -0.5$, $\varepsilon = 0.5$

Mode	α	β	μ = -0.5					μ = 0.0					μ = 1.0				
			*	Ωc	Ωs			*	Ωc	Ωs			*	Ωc	Ωs		
				h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2
I	-0.5	0	10.8494	0.3132	0.1559	0.3075	0.5843	12.6628	0.3655	0.1819	0.3590	0.6828	17.2232	0.4972	0.2475	0.4885	0.9303
		0.5	11.9438	0.3448	0.1711	0.3348	0.6197	13.9321	0.4022	0.1996	0.3907	0.7241	18.9623	0.5474	0.2717	0.5322	0.9878
	0	-0.5	13.1438	0.3794	0.1885	0.3700	0.6914	15.3372	0.4427	0.2200	0.4319	0.8079	20.8457	0.6018	0.2990	0.5875	1.1007
		0	14.0042	0.4043	0.2002	0.3895	0.7083	16.3273	0.4713	0.2335	0.4545	0.8274	22.1885	0.6405	0.3173	0.6182	1.1279
		0.5	15.3582	0.4434	0.2187	0.4211	0.7432	17.9162	0.5172	0.2552	0.4916	0.8689	24.4078	0.7046	0.3477	0.6704	1.1871
	0.5	-0.5	16.1238	0.4655	0.2300	0.4446	0.7933	18.7929	0.5425	0.2681	0.5186	0.9268	25.5145	0.7365	0.3641	0.7050	1.2630
		0	17.3482	0.5008	0.2464	0.4711	0.8160	20.2259	0.5839	0.2873	0.5497	0.9538	27.5087	0.7941	0.3910	0.7487	1.3020
		0.5	18.8032	0.5428	0.2657	0.5018	0.8434	21.9396	0.6333	0.3101	0.5861	0.9866	29.9169	0.8636	0.4231	0.8003	1.3496
II	-0.5	0	58.0524	1.6758	0.8176	1.5296	2.5050	69.3651	2.0024	0.9764	1.8243	2.9768	98.8334	2.8531	1.3894	2.5874	4.1871
		0.5	77.6192	2.2407	1.0672	1.8896	2.7648	92.4712	2.6694	1.2704	2.2452	3.2721	130.9857	3.7812	1.7960	3.1599	4.5647
	0	-0.5	67.3097	1.9431	0.9413	1.7316	2.7350	80.4959	2.3237	1.1249	2.0655	3.2483	114.8827	3.3164	1.6026	2.9297	4.5622
		0	87.0329	2.5124	1.1852	2.0584	2.9231	103.7642	2.9954	1.4116	2.4458	3.4581	147.1940	4.2491	1.9974	3.4415	4.8189
		0.5	105.9773	3.0593	1.3978	2.2875	2.9637	126.1499	3.6416	1.6617	2.7118	3.4987	178.3925	5.1497	2.3421	3.7977	4.8571
	0.5	-0.5	96.4173	2.7833	1.2997	2.2137	3.0606	115.0288	3.3206	1.5485	2.6302	3.6193	163.3814	4.7164	2.1925	3.6994	5.0387
		0	115.5140	3.3346	1.5052	2.4163	3.0693	137.5828	3.9717	1.7899	2.8641	3.6222	194.7787	5.6228	2.5238	4.0089	5.0248
		0.5	134.1809	3.8735	1.6833	2.5512	3.0080	159.6462	4.6086	1.9989	3.0189	3.5464	225.5475	6.5110	2.8104	4.2115	4.9129
III	-0.5	0	156.0734	4.5055	2.1326	3.7389	5.4820	187.8198	5.4219	2.5638	4.4858	6.5607	271.5179	7.8380	3.6975	6.4389	9.3648
		0.5	222.8635	6.4335	2.8933	4.6581	6.1213	267.5458	7.7234	3.4696	5.5763	7.3228	384.8813	11.1106	4.9770	7.9652	10.4403
	0	-0.5	177.6417	5.1281	2.3925	4.0840	5.7832	213.9461	6.1761	2.8775	4.8995	6.9220	309.7804	8.9426	4.1529	7.0305	9.8832
		0	245.5679	7.0889	3.1283	4.9091	6.3084	295.0015	8.5160	3.7524	5.8766	7.5484	424.9537	12.2674	5.3850	8.3931	10.7672
		0.5	310.1313	8.9527	3.7168	5.4363	6.4220	372.0848	10.7412	4.4527	6.5032	7.6988	534.6079	15.4328	6.3720	9.2731	11.0105
	0.5	-0.5	267.9911	7.7362	3.3479	5.1300	6.4721	322.1291	9.2991	4.0164	6.1408	7.7464	464.5820	13.4113	5.7654	8.7695	11.0553
		0	333.3273	9.6223	3.9080	5.6019	6.5205	400.1218	11.5505	4.6822	6.7015	7.8202	575.4899	16.6130	6.7013	9.5564	11.1912
		0.5	396.9417	11.4587	4.3598	5.9090	6.2258	476.0778	13.7432	5.2196	7.0684	7.5042	683.5581	19.7326	7.4576	10.0766	10.8288

* for general value of h_0

Table 4.17

Values of frequency parameter Ω for C-F plate for $\eta = 0.0$, $\varepsilon = 0.5$

Mode	α	β	$\mu = -0.5$					$\mu = 0.0$					$\mu = 1.0$				
			*	Ωc	Ωs			*	Ωc	Ωs			*	Ωc	Ωs		
				h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2
I	-0.5	0	8.6541	0.2498	0.1243	0.2454	0.4668	10.1035	0.2917	0.1452	0.2866	0.5456	13.7499	0.3969	0.1976	0.3902	0.7439
		0.5	9.5219	0.2749	0.1364	0.2671	0.4950	11.1095	0.3207	0.1592	0.3118	0.5785	15.1269	0.4367	0.2168	0.4248	0.7896
	0	-0.5	10.4907	0.3028	0.1505	0.2955	0.5530	12.2453	0.3535	0.1757	0.3451	0.6464	16.6532	0.4807	0.2389	0.4696	0.8813
		0	11.1686	0.3224	0.1597	0.3109	0.5663	13.0243	0.3760	0.1863	0.3628	0.6617	17.7076	0.5112	0.2533	0.4938	0.9025
		0.5	12.2452	0.3535	0.1744	0.3360	0.5943	14.2875	0.4124	0.2036	0.3924	0.6950	19.4718	0.5621	0.2775	0.5354	0.9501
	0.5	-0.5	12.8632	0.3713	0.1835	0.3550	0.6348	14.9963	0.4329	0.2140	0.4143	0.7419	20.3695	0.5880	0.2908	0.5634	1.0116
		0	13.8350	0.3994	0.1966	0.3761	0.6529	16.1333	0.4657	0.2293	0.4390	0.7635	21.9513	0.6337	0.3121	0.5982	1.0429
		0.5	14.9927	0.4328	0.2120	0.4006	0.6751	17.4969	0.5051	0.2474	0.4681	0.7900	23.8675	0.6890	0.3377	0.6394	1.0813
II	-0.5	0	47.4405	1.3695	0.6684	1.2519	2.0556	56.7240	1.6375	0.7988	1.4943	2.4453	80.9326	2.3363	1.1384	2.1230	3.4467
		0.5	63.6057	1.8361	0.8749	1.5504	2.2707	75.8262	2.1889	1.0422	1.8438	2.6902	107.5479	3.1046	1.4756	2.5996	3.7607
	0	-0.5	54.9651	1.5867	0.7692	1.4171	2.2455	65.7788	1.8989	0.9199	1.6918	2.6700	94.0101	2.7138	1.3125	2.4040	3.7585
		0	71.2740	2.0575	0.9712	1.6889	2.4019	85.0328	2.4547	1.1577	2.0087	2.8446	120.7835	3.4867	1.6405	2.8317	3.9725
		0.5	86.9135	2.5090	1.1470	1.8783	2.4312	103.5248	2.9885	1.3645	2.2289	2.8731	146.5860	4.2316	1.9262	3.1272	3.9967
	0.5	-0.5	78.9140	2.2781	1.0647	1.8166	2.5160	94.2111	2.7196	1.2696	2.1604	2.9787	133.9947	3.8681	1.8004	3.0446	4.1560
		0	94.6871	2.7334	1.2348	1.9844	2.5189	112.8516	3.2577	1.4695	2.3545	2.9759	159.9757	4.6181	2.0754	3.3020	4.1366
		0.5	110.0930	3.1781	1.3819	2.0949	2.4618	131.0717	3.7837	1.6423	2.4815	2.9055	185.4137	5.3524	2.3128	3.4685	4.0328
III	-0.5	0	128.4936	3.7093	1.7573	3.0866	4.5369	154.7432	4.4671	2.1144	3.7070	5.4363	224.0309	6.4672	3.0545	5.3321	7.7787
		0.5	184.1402	5.3157	2.3926	3.8569	5.0683	221.2246	6.3862	2.8718	4.6226	6.0713	318.7246	9.2008	4.1272	6.6183	8.6791
	0	-0.5	146.0998	4.2175	1.9702	3.3710	4.7858	176.0865	5.0832	2.3716	4.0485	5.7353	255.3381	7.3710	3.4287	5.8220	8.2086
		0	202.7064	5.8516	2.5854	4.0639	5.2218	243.6940	7.0348	3.1042	4.8707	6.2567	351.5726	10.1490	4.4634	6.9730	8.9484
		0.5	256.4989	7.4045	3.0771	4.5043	5.2955	307.9721	8.8904	3.6901	5.3951	6.3603	443.1647	12.7931	5.2919	7.7125	9.1278
	0.5	-0.5	221.0341	6.3807	2.7658	4.2463	5.3560	265.8844	7.6754	3.3214	5.0892	6.4193	384.0407	11.0863	4.7771	7.2853	9.1855
		0	275.4763	7.9523	3.2342	4.6408	5.3730	330.9284	9.5531	3.8791	5.5588	6.4566	476.6929	13.7609	5.5638	7.9470	9.2731
		0.5	328.4794	9.4824	3.6117	4.8943	5.1048	394.2673	11.3815	4.3288	5.8626	6.1669	566.9603	16.3667	6.1989	8.3798	8.9381

* for general value of h_0

Table 4.18

Values of frequency parameter Ω for C-F plate for $\eta = 1.0, \varepsilon = 0.5$

Mode	α	β	$\mu = -0.5$					$\mu = 0.0$					$\mu = 1.0$				
			*	Ωc	Ωs			*	Ωc	Ωs			*	Ωc	Ωs		
				h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2
I	-0.5	0	5.4937	0.1586	0.0789	0.1559	0.2971	6.4174	0.1853	0.0922	0.1821	0.3475	8.7426	0.2524	0.1257	0.2483	0.4744
		0.5	6.0387	0.1743	0.0865	0.1696	0.3150	7.0483	0.2035	0.1010	0.1980	0.3683	9.6047	0.2773	0.1377	0.2700	0.5032
	0	-0.5	6.6676	0.1925	0.0957	0.1880	0.3527	7.7873	0.2248	0.1117	0.2197	0.4126	10.6025	0.3061	0.1522	0.2993	0.5633
		0	7.0876	0.2046	0.1014	0.1976	0.3609	8.2689	0.2387	0.1183	0.2306	0.4220	11.2516	0.3248	0.1610	0.3142	0.5762
		0.5	7.7671	0.2242	0.1107	0.2135	0.3789	9.0659	0.2617	0.1292	0.2494	0.4433	12.3643	0.3569	0.1763	0.3406	0.6067
	0.5	-0.5	8.1683	0.2358	0.1166	0.2258	0.4053	9.5272	0.2750	0.1360	0.2636	0.4739	12.9522	0.3739	0.1850	0.3589	0.6470
		0	8.7793	0.2534	0.1248	0.2392	0.4169	10.2418	0.2957	0.1456	0.2793	0.4877	13.9457	0.4026	0.1984	0.3809	0.6670
		0.5	9.5109	0.2746	0.1346	0.2548	0.4312	11.1034	0.3205	0.1571	0.2978	0.5048	15.1564	0.4375	0.2146	0.4071	0.6918
II	-0.5	0	31.6144	0.9126	0.4458	0.8365	1.3794	37.8524	1.0927	0.5335	1.0000	1.6444	54.1542	1.5633	0.7625	1.4255	2.3275
		0.5	42.6180	1.2303	0.5865	1.0406	1.5252	50.8728	1.4686	0.6997	1.2397	1.8108	72.3428	2.0884	0.9936	1.7538	2.5418
	0	-0.5	36.5756	1.0558	0.5124	0.9465	1.5083	43.8320	1.2653	0.6137	1.1320	1.7975	62.8188	1.8134	0.8783	1.6142	2.5415
		0	47.6958	1.3769	0.6506	1.1335	1.6148	56.9791	1.6448	0.7766	1.3506	1.9167	81.1501	2.3426	1.1039	1.9109	2.6881
		0.5	58.3272	1.6838	0.7701	1.2619	1.6279	69.5652	2.0082	0.9176	1.5003	1.9279	98.7536	2.8508	1.2993	2.1128	2.6927
	0.5	-0.5	52.7490	1.5227	0.7127	1.2193	1.6931	63.0596	1.8204	0.8512	1.4529	2.0089	89.9312	2.5961	1.2108	2.0554	2.8153
		0	63.4811	1.8325	0.8287	1.3335	1.6881	75.7591	2.1870	0.9878	1.5854	1.9987	107.6749	3.1083	1.3995	2.2320	2.7898
		0.5	73.9480	2.1347	0.9286	1.4069	1.6400	88.1529	2.5448	1.1054	1.6698	1.9395	125.0182	3.6090	1.5616	2.3430	2.7027
III	-0.5	0	86.9333	2.5095	1.1907	2.0977	3.0967	104.8462	3.0266	1.4350	2.5246	3.7198	152.2413	4.3948	2.0801	3.6465	5.3489
		0.5	125.4815	3.6223	1.6322	2.6360	3.4606	150.9803	4.3584	1.9628	3.1667	4.1572	218.1826	6.2984	2.8315	4.5552	5.9755
	0	-0.5	98.6394	2.8475	1.3330	2.2899	3.2660	119.0589	3.4369	1.6073	2.7561	3.9237	173.1547	4.9985	2.3320	3.9808	5.6434
		0	137.8683	3.9799	1.7616	2.7760	3.5633	165.9950	4.7919	2.1192	3.3352	4.2816	240.2033	6.9341	3.0589	4.7977	6.1573
		0.5	175.1374	5.0558	2.1033	3.0813	3.5805	210.6048	6.0796	2.5277	3.7003	4.3181	303.9868	8.7753	3.6401	5.3168	6.2435
	0.5	-0.5	150.0848	4.3326	1.8828	2.8997	3.6529	180.8094	5.2195	2.2655	3.4839	4.3905	261.9487	7.5618	3.2714	5.0116	6.3172
		0	187.8102	5.4216	2.2091	3.1735	3.6269	225.9592	6.5229	2.6552	3.8113	4.3770	326.4852	9.4248	3.8248	5.4768	6.3355
		0.5	224.5355	6.4818	2.4712	3.3446	3.4154	269.9204	7.7919	2.9686	4.0176	4.1444	389.3489	11.2395	4.2706	5.7741	6.0599

* for general value of h_0

Table 4.19

Comparison of frequency parameter Ω for homogeneous (μ=0.0, η=0.0) uniform thickness(α=0.0, β=0.0) Mindlin annular plates

Mode \ h_0	ε = 0.3				ε = 0.5		
	0.1	0.2	0.3		0.1	0.2	0.3
C-C							
I	39.3972 39.40*	30.0407 30.04*	23.1321 23.13*		70.2762 70.28*	48.3104 48.31*	35.3165 35.32*
II	95.5919 95.59*	64.2314 64.23*	46.6591 46.66*		159.7852 159.78*	97.3880 97.39*	68.2971 68.30*
C-S							
I	27.3803 27.38*	22.4405 22.44*	18.1149 18.12*		51.2202 51.22*	38.3632 38.36*	29.2636 29.26*
II	82.1720 82.17*	59.3834 59.38*	45.0414 45.04*		142.7106 142.71*	93.7802 93.78*	68.0777 68.08*
C-F							
I	6.5160 6.52*	6.1367 6.14*	5.6361 5.64*		12.5678 12.57*	11.4610 11.46*	10.1567 10.16*
II	37.8938 37.89*	29.7618 29.76*	23.2707 23.27*		69.5834 69.58*	49.2699 49.27*	36.1560 36.16*

* Values taken from Irie et al.[1982].

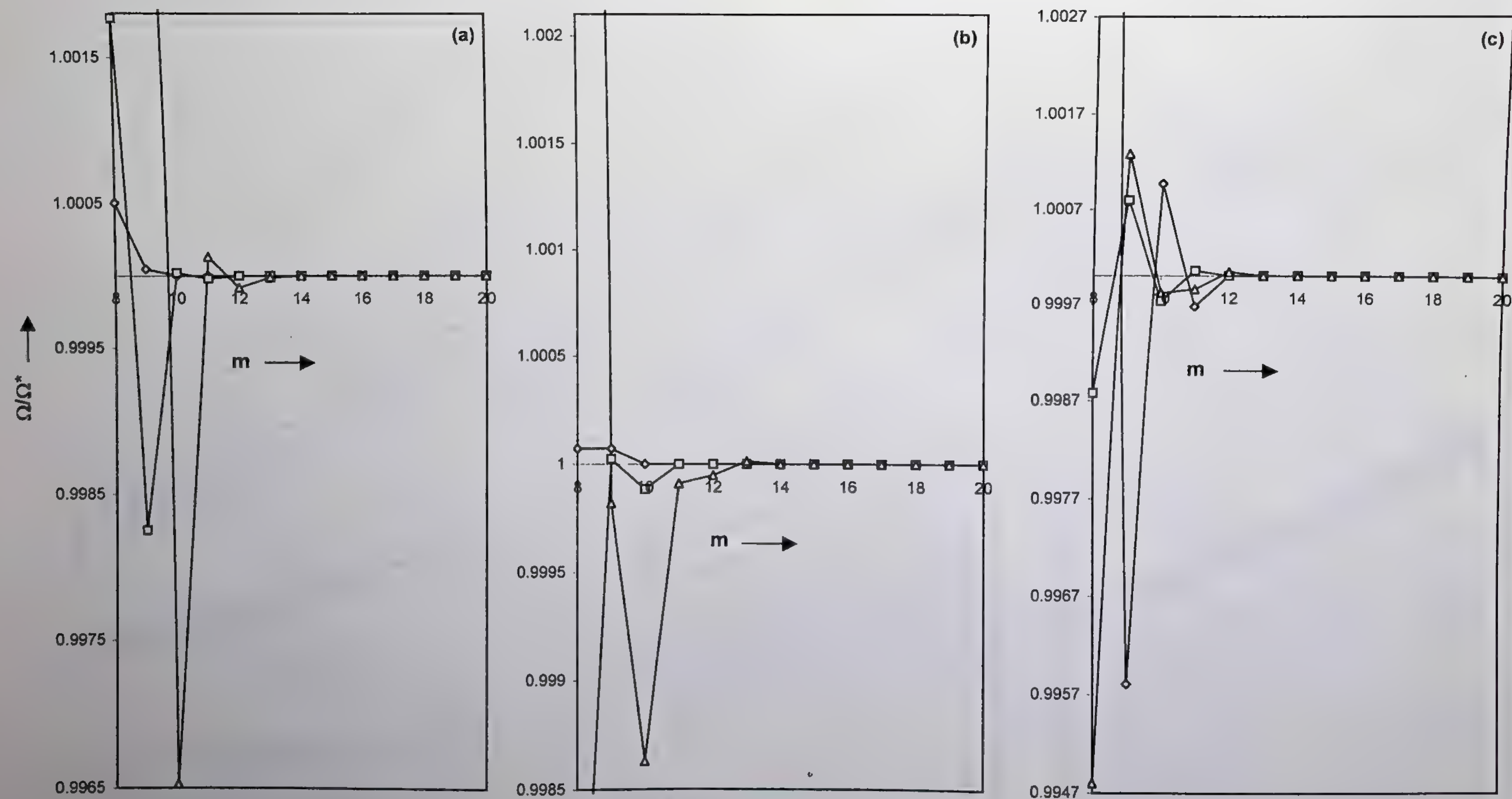

Fig. 4.1 : Convergence of the Normalized Frequency Parameter Ω/Ω^* for the first three modes of vibration with grid refinement for $\eta = -0.5$, $\mu = 1.0$, $\alpha = 0.0$, $\beta = 0.5$, $\varepsilon = 0.3$, $h_0 = 0.1$ for (a) C-C (b) C-S and (c) C-F plate.

——◇——, Fundamental mode; ——□——, Second mode; ——Δ——, Third mode.

Ω^*- the corresponding results using 20 collocation points.

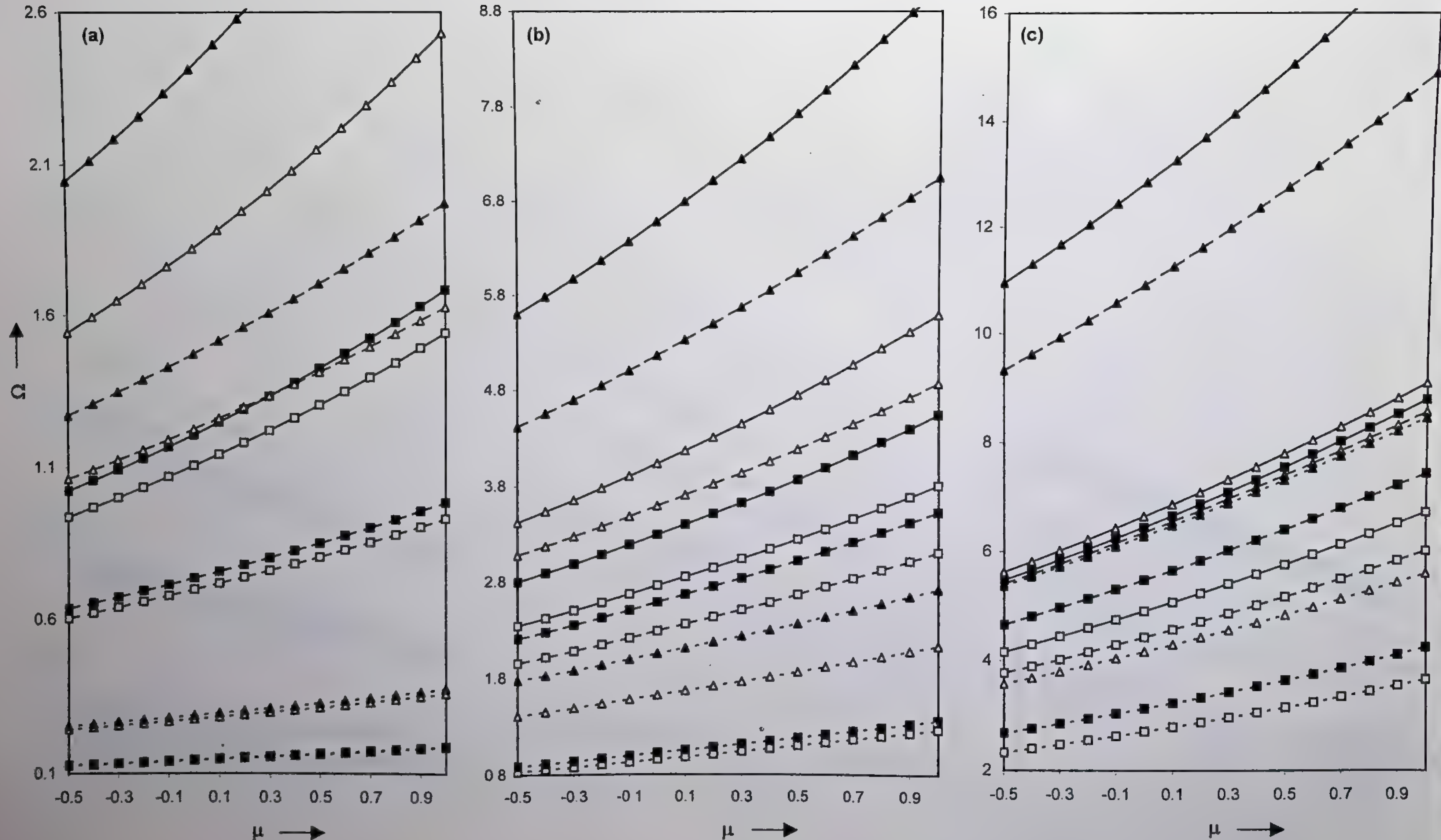

Fig. 4.2 : Frequency parameter for C-C, C-S and C-F plates vibrating in (a) fundamental (b) second and (c) third mode for $\eta = -0.5$, $\alpha = 0.5$, $\beta = 0.5$, $\varepsilon = 0.3$. ————, C-C ; – – – – – –, C-S ; -----------, C-F.

□ $h_0 = 0.05$; Δ $h_0 = 0.1$, □ Δ shear plate theory ; ■ ▲ classical plate theory

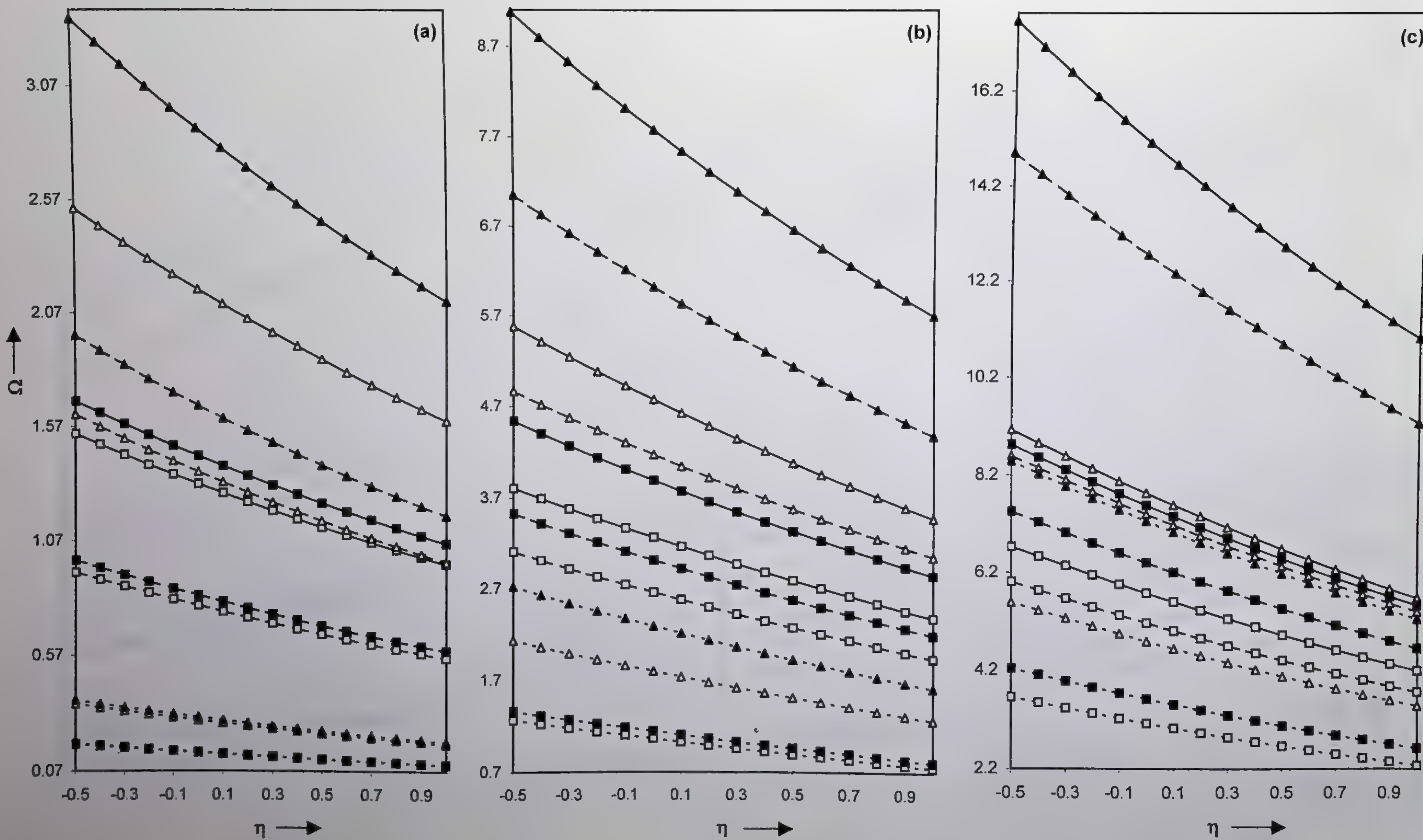

Fig. 4.3 : Frequency parameter for C-C, C-S and C-F plates vibrating in (a) fundamental (b) second and (c) third mode for $\mu = 1.0$, $\alpha = 0.5$, $\beta = 0.5$, $\varepsilon = 0.3$. ————, C-C ; – – – – – –, C-S ; -----------, C-F.
□, $h_0 = 0.05$; Δ, $h_0 = 0.1$. □, Δ, shear plate theory ; ■, ▲, classical plate theory

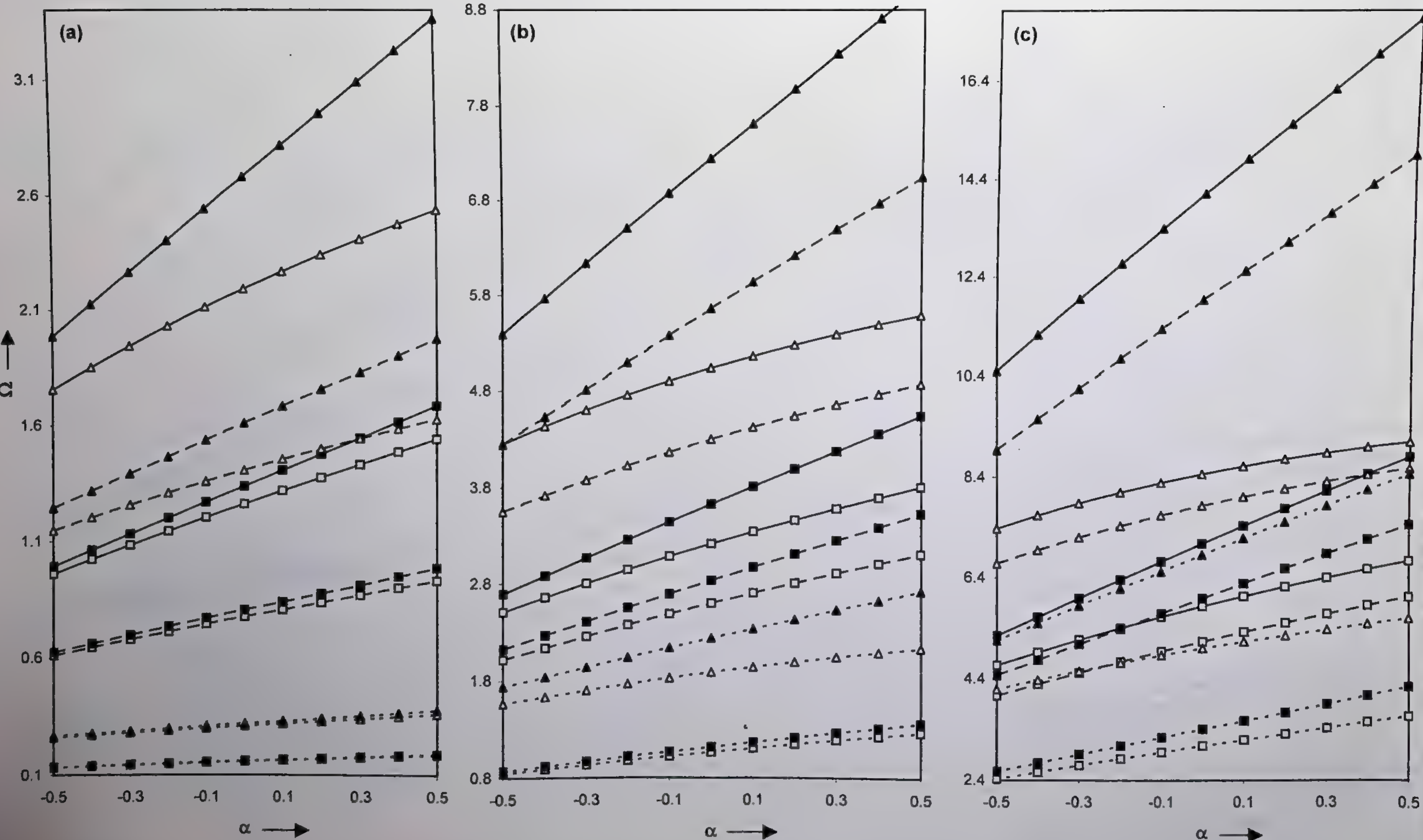

Fig. 4.4 : Frequency parameter for C-C, C-S and C-F plates vibrating in (a) fundamental (b) second and (c) third mode for $\mu = 1.0$, $\eta = -0.5$, $\beta = 0.5$, $\varepsilon = 0.3$. ———, C-C ; – – – – – –, C-S ; ··········, C-F.

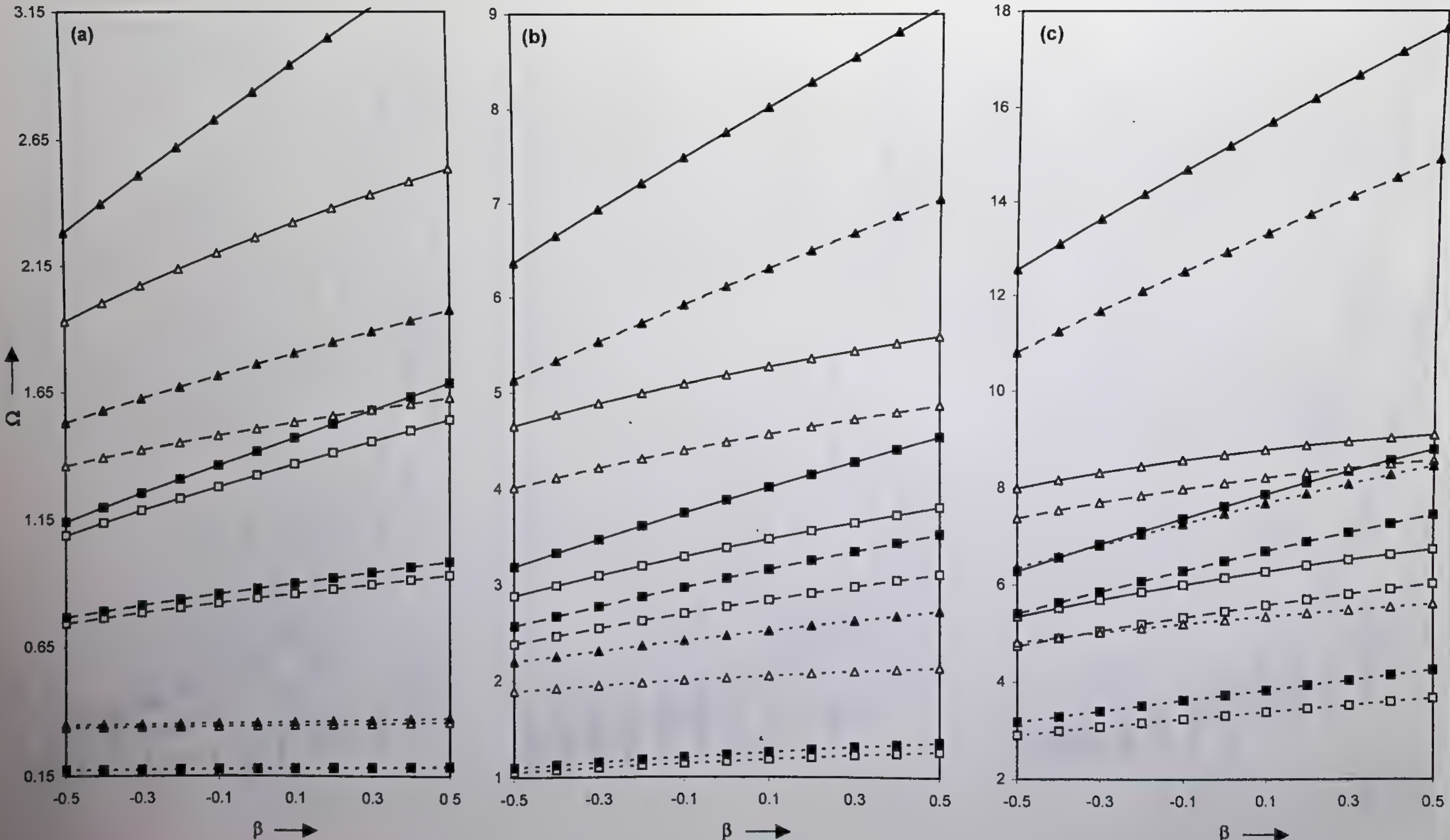

Fig. 4.5 : Frequency parameter for C-C, C-S and C-F plates vibrating in (a) fundamental (b) second and (c) third mode for $\mu = 1.0$, $\eta = -0.5$, $\alpha = 0.5$, $\varepsilon = 0.3$. ———, C-C ; – – – – – –, C-S ; -----------, C-F.

□, $h_0 = 0.05$; Δ, $h_0 = 0.1$. □, Δ, shear plate theory ; ■, ▲, classical plate theory.

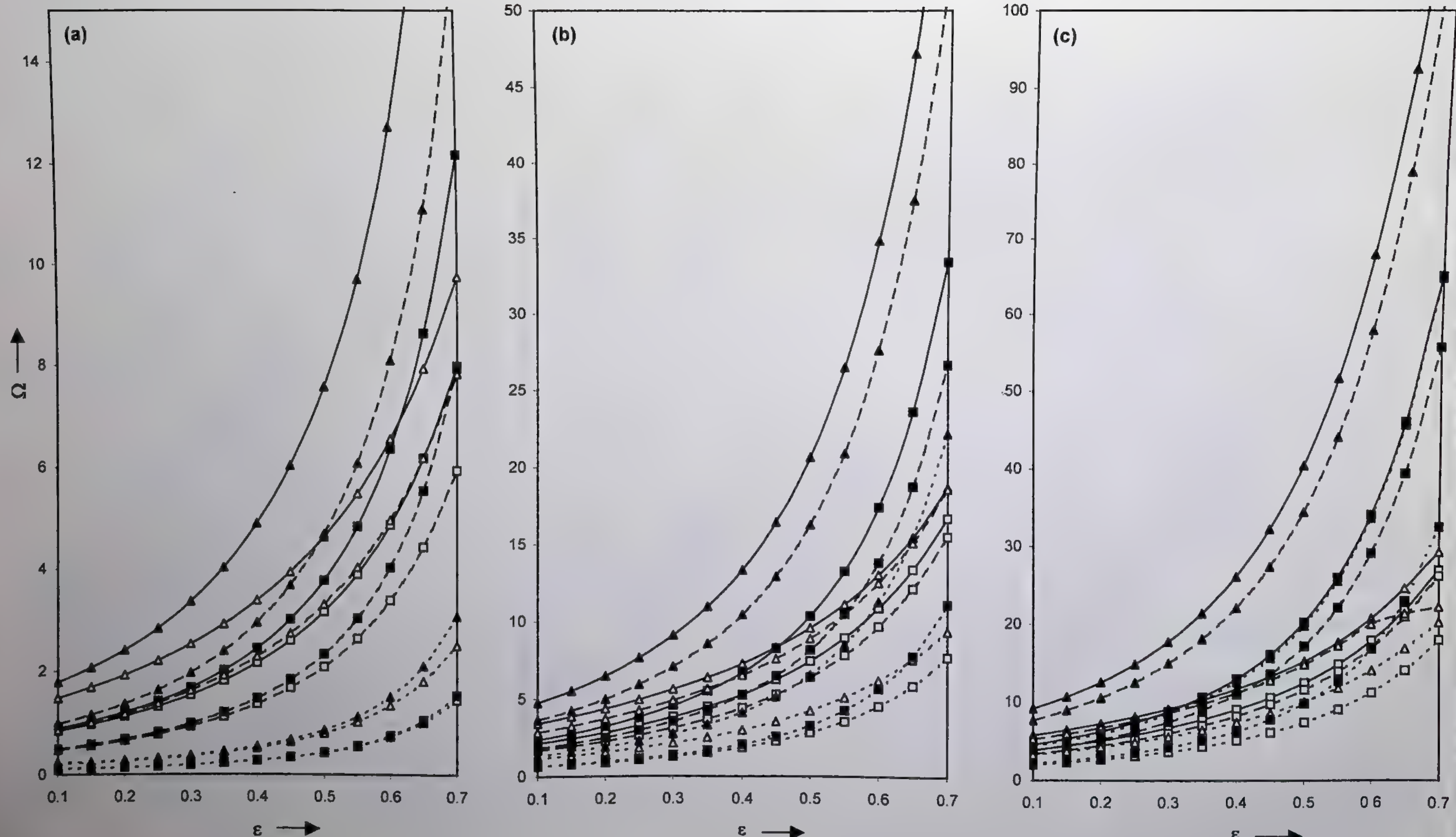

Fig. 4.6 : Frequency parameter for C-C, C-S and C-F plates vibrating in (a) fundamental (b) second and (c) third mode for $\mu = 1.0$, $\eta = -0.5$, $\alpha = 0.5$, $\beta = 0.5$. ————, C-C ; – – – – – –, C-S ; -----------, C-F.

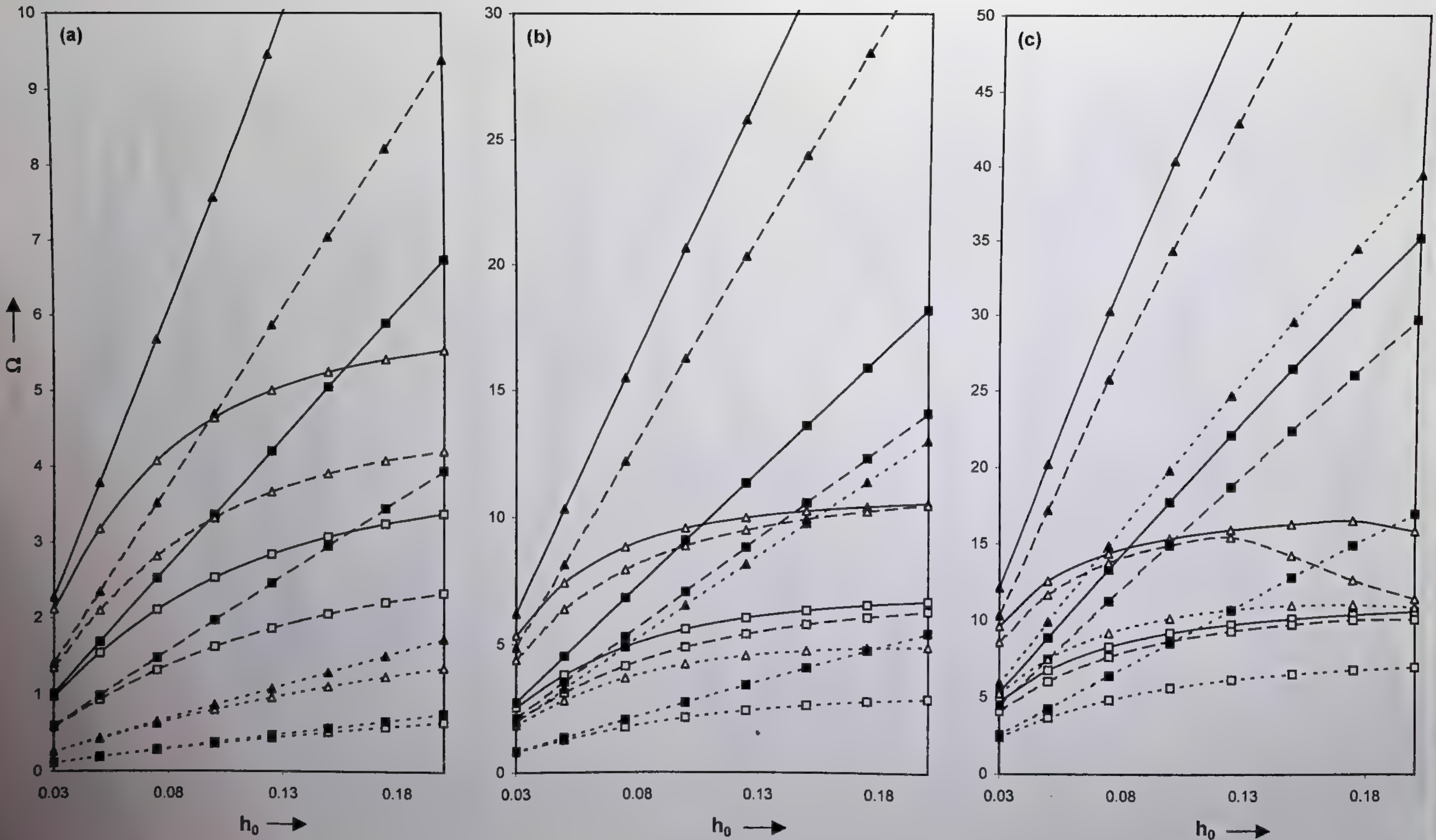

Fig. 4.7 : Frequency parameter for C-C, C-S and C-F plates vibrating in (a) fundamental (b) second and (c) third mode for $\mu = 1.0$, $\eta = -0.5$, $\alpha = 0.5$, $\beta = 0.5$. ————, C-C ; – – – – – –, C-S ; -----------, C-F.
□, $\varepsilon = 0.3$; Δ, $\varepsilon = 0.5$. □, Δ, shear plate theory ; ■, ▲, classical plate theory.

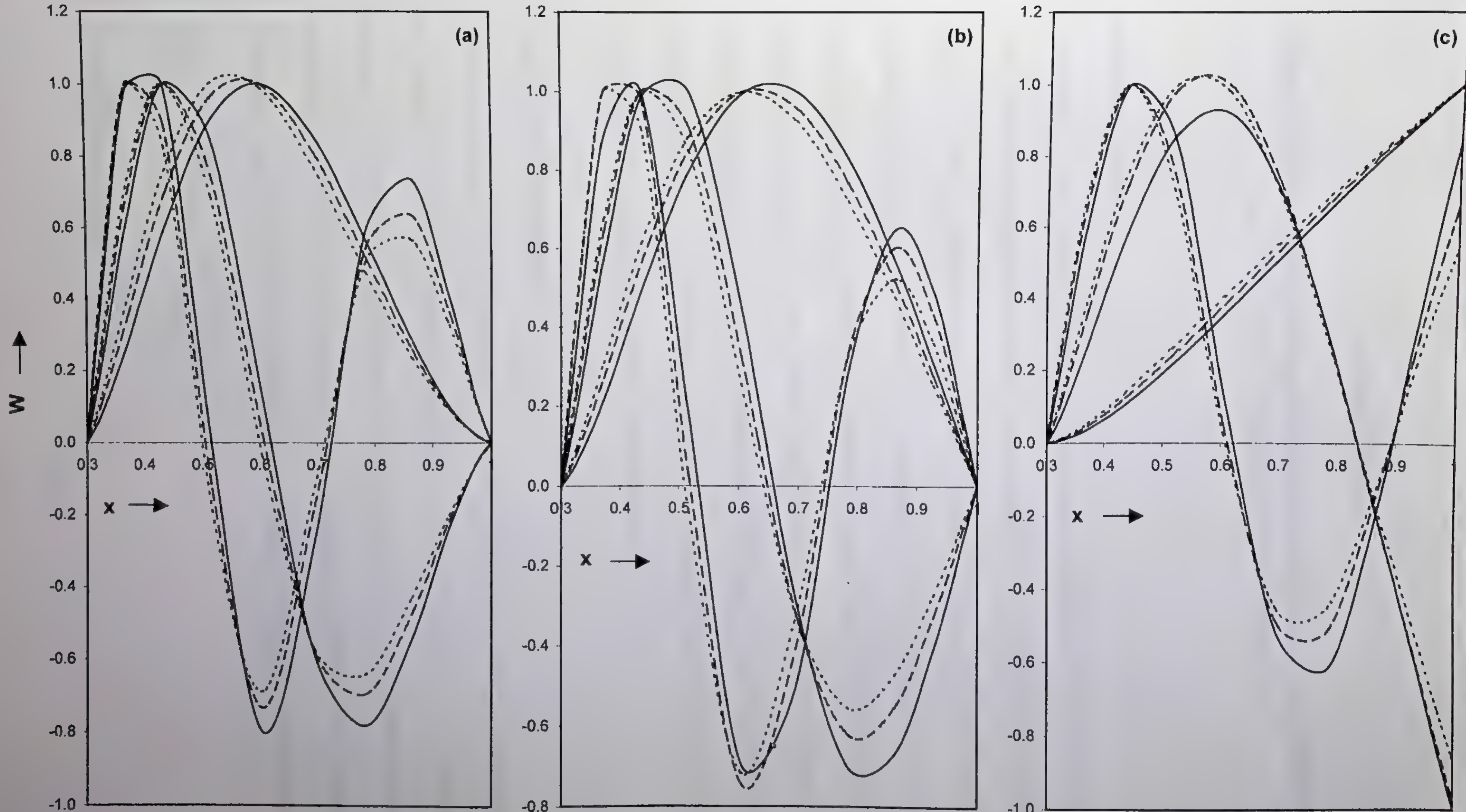

Fig. 4.8 : Normalized displacements for the first three modes of vibration for (a) C-C (b) C-S and (c) C-F plates for $\mu = 1.0$, $\eta = -0.5$ and $h_0 = 0.1$.

$\alpha = 0.0, \beta = 0.0$; $\alpha = 0.5, \beta = 0.0$; $\alpha = 0.5, \beta = 0.5$

CHAPTER V

VIBRATIONS OF NON-HOMOGENEOUS CIRCULAR MINDLIN PLATES WITH VARIABLE THICKNESS

1. INTRODUCTION

This chapter has been devoted to investigate the natural frequencies of non-linearly tapered circular plates of moderate thickness taking into account the non-homogeneity which arises due to variation in Young's modulus and density of the plate material, using the first order shear deformation plate theory of Mindlin. The consideration of non-homogeneity and variable thickness together with the inclusion of transverse shear and rotatory inertia leads to a set of coupled differential equations with variable coefficients. An approximate solution has been obtained employing Chebyshev collocation technique. First three natural frequencies have been computed for various values of plate parameters for three different edge conditions. Mode shapes for specified plate parameters have also been presented.

2. EQUATION OF MOTION

Consider an isotropic non-homogeneous moderately thick circular plate of radius a, thickness $h(r)$, density $\rho(r)$ referred to a cylindrical polar coordinate system (r, θ, z). The differential equations, which govern the axisymmetric motion of such plates, have been derived according to Mindlin's plate theory[1955], using Hamilton's principle as in chapter IV by taking limits of integration in equations (4.2.6- 4.2.17) from 0 to a instead of b to a and are given below:

$$\frac{\partial M_r}{\partial r}+\frac{M_r-M_\theta}{r}-Q_r-\frac{\rho h^3}{12}\frac{\partial^2\psi_r}{\partial t^2}=0, \tag{5.2.1}$$

$$\frac{1}{r}Q_r+\frac{\partial Q_r}{\partial r}-\rho h\frac{\partial^2 w}{\partial t^2}=0, \tag{5.2.2}$$

where t is the time, w the transverse deflection, ψ_r the angle of rotation in the rz-plane and M_r, M_θ and Q_r are the moment and shear resultants all per unit length given by

$$M_r = D\left(\frac{\partial \psi_r}{\partial r} + \frac{\upsilon}{r}\psi_r\right),$$

$$M_\theta = D\left(\frac{\psi_r}{r} + \upsilon\frac{\partial \psi_r}{\partial r}\right), \tag{5.2.3}$$

$$Q_r = \kappa G h\left(\psi_r + \frac{\partial w}{\partial r}\right),$$

where $D(\equiv D(r)) = \dfrac{E(r)h^3(r)}{12(1-\upsilon^2)}$ is the flexural rigidity, $\kappa\left(=\dfrac{\pi^2}{12}\right)$ an averaging shear coefficient and $E(r), G(\equiv G(r)), \upsilon$ are the elastic constants.

Introducing non-dimensional variables

$$R = r/a, \quad H = h/a, \quad \bar{w} = w/a, \quad T = t\sqrt{E_0/\rho_0 a^2(1-\upsilon^2)} \tag{5.2.4}$$

together with quadratic thickness variation i.e.

$$H = h_0(1+\alpha R + \beta R^2) \text{ such that } |\alpha| \le 1, |\beta| \le 1 \text{ and } \alpha + \beta > -1, \tag{5.2.5}$$

and assuming the exponential variation for the non-homogeneity of material as follows :

$$E = E_0 e^{\mu R}, \qquad \rho = \rho_0 e^{\eta R}, \tag{5.2.6}$$

equations (5.2.1) and (5.2.2) reduce to

$$A_1\frac{dW}{dR} + A_2\frac{d^2\psi}{dR^2} + A_3\frac{d\psi}{dR} + (A_4 + A_5\Omega^2)\psi = 0, \tag{5.2.7}$$

$$B_1\frac{d^2W}{dR^2} + B_2\frac{dW}{dR} + B_3\Omega^2 W + B_4\frac{d\psi}{dR} + B_5\psi = 0 \tag{5.2.8}$$

where $\overline{w}(R,T) = W(R)e^{i\Omega T}$, $\psi_r(R,T) = \psi(R)e^{i\Omega T}$ (for harmonic vibrations), Ω is the frequency parameter, μ and η are non-homogeneity parameters, α and β are taper parameters, h_0, ρ_0 and E_0 are thickness, density and Young's modulus respectively, at the centre of the plate. The coefficients A_i and B_i, $i = 1, 2, 3, 4, 5$ are the same as given by relations (4.2.31).

Coupled differential equations (5.2.7) and (5.2.8), together with edge conditions at the edge $R = 1$ and regularity condition at centre of plate $R = 0$ constitute a boundary value problem, which has been solved by Chebyshev collocation technique. The present technique is preferred because Chebyshev polynomials have minimax property, i.e. of all the monic polynomials, the maximum error is minimum (Fox and Parker[1968], Snyder[1969]).

3. METHOD OF SOLUTION :CHEBYSHEV COLLOCATION TECHNIQUE

By taking a new independent variable

$$x \equiv 2R - 1 \quad , \tag{5.3.1}$$

the range $0 \le R \le 1$ gets transformed to $-1 \le x \le 1$ which is the applicability range of the Chebyshev collocation technique and equations (5.2.7) and (5.2.8) now reduce to

$$U_1 \frac{dW}{dx} + U_2 \frac{d^2\psi}{dx^2} + U_3 \frac{d\psi}{dx} + (U_4 + \Omega^2 U_5)\psi = 0, \tag{5.3.2}$$

$$V_1 \frac{d^2W}{dx^2} + V_2 \frac{dW}{dx} + V_3\Omega^2 W + V_4 \frac{d\psi}{dx} + V_5\psi = 0, \tag{5.3.3}$$

where

$U_1 = 2A_1$, $U_2 = 4A_2$, $U_3 = 2A_3$, $U_4 = A_4$, $U_5 = A_5$, and

$V_1 = 4B_1$, $V_2 = 2B_2$, $V_3 = B_3$, $V_4 = 2B_4$, $V_5 = B_5$.

According to Chebyshev Collocation technique (Lal and Gupta[1982]), we assume

$$\frac{d^2W}{dx^2} = \sum_{k=0}^{m-3} a_{k+3}T_k \qquad \text{and} \tag{5.3.4}$$

$$\frac{d^2\psi}{dx^2} = \sum_{k=0}^{m-3} b_{k+3}T_k \ , \tag{5.3.5}$$

where a_j and b_j (j = 3, 4,…, m) are the unknown constants and T_j (j = 0, 1, 2,…, m-3) are the Chebyshev polynomials.

Successive integration of eqs.(5.3.4) and (5.3.5) leads to

$$W = a_1 + a_2T_1 + \sum_{k=0}^{m-3} a_{k+3}T_k^2 \qquad \text{and} \tag{5.3.6}$$

$$\psi = b_1 + b_2T_1 + \sum_{k=0}^{m-3} b_{k+3}T_k^2 \ , \tag{5.3.7}$$

where a_1 , a_2 , b_1 and b_2 are the constants of integration and $T_k^{\prime}$ represents the j^{th} integral of T_k which are defined as

$$T_0^1 = T_1 \ ; \qquad\qquad T_1^1 = \frac{1}{4}(T_2 + T_o) \ ;$$

$$T_j^1 = \int T_j dx = \frac{1}{2}\left[\frac{T_{j+1}}{(j+1)} - \frac{T_{j-1}}{(j-1)}\right], \ j > 1 \ ;$$

$$T_j^i = \int T^{i-1}_j \, dx \ ; \qquad T_j = 2xT_{j-1} - T_{j-2} \, , \qquad j \geq 2 \ ;$$

$$T_1 = x, \qquad T_0 = 1.$$

Substitution of W, ψ and their derivatives in equations (5.3.2) and (5.3.3) gives simultaneous equations in terms of the T's, a's and b's. The satisfaction of this resultant set of equations at (m-2) collocation points given by

$$x_i = \cos\left(\frac{(2i-1)}{(m-2)}\cdot\frac{\pi}{2}\right), \quad i = 1,2,\ldots\ldots, m-2, \tag{5.3.8}$$

provides a set of 2(m-2) equations in unknowns a_j and b_j (j=1, 2,…., m), which can be written in matrix form as

$$[B][C^*] = [0] \ , \tag{5.3.9}$$

where B and C^* are matrices of order (2m-4) x 2m and 2m x 1, respectively.

4. BOUNDARY CONDITIONS AND FREQUENCY EQUATIONS

By satisfying the relations for,

(1) clamped edge : $\quad W = \psi = 0;$

(2) simply supported edge : $\quad W = \dfrac{\partial\psi}{\partial R} + \dfrac{\upsilon}{R}\psi = 0;$

(3) free edge : $\quad \psi + \dfrac{\partial W}{\partial R} = \dfrac{\partial\psi}{\partial R} + \dfrac{\upsilon}{R}\psi = 0;$

at the boundary of the plate, along with

(4) regularity condition : $\quad \psi = Q_r = 0;$

at the centre of the plate, a set of four homogeneous equations are obtained. For a clamped plate, these equations together with the field equations (5.3.9) can be written as

$$\begin{bmatrix} B \\ B^C \end{bmatrix}[C^*] = [0], \tag{5.4.1}$$

where B^C is a matrix of order 4 x 2m.

For a non-trivial solution of equation (5.4.1), the frequency determinant must vanish and hence,

$$\begin{vmatrix} B \\ B^{C} \end{vmatrix} = 0. \tag{5.4.2}$$

Similarly, for simply supported and free plates, frequency determinants can respectively be written as

$$\begin{vmatrix} B \\ B^{S} \end{vmatrix} = 0, \qquad \begin{vmatrix} B \\ B^{F} \end{vmatrix} = 0 \quad . \tag{5.4.3, 5.4.4}$$

5. NUMERICAL RESULTS AND DISCUSSION

The frequency equations (5.4.2-5.4.4) provide the values of the frequency parameter Ω for various values of plate parameters. Numerical computation has been carried out to investigate the effect of non-homogeneity parameter μ = -0.5(0.1)1.0, density parameter η = -0.5(0.1)1.0. taper parameters α = -0.5(0.1)0.5 and β = -0.5(0.1)0.5 (such that $\alpha + \beta > -1$). thickness parameter h_0 = 0.05(0.05)0.2 on first three natural frequencies by shear plate theory of Mindlin (SPT) and classical plate theory (CPT) for υ = 0.3. The results on the basis of classical plate theory have been obtained by eliminating Q_r from equations (5.2.1) and (5.2.2) after neglecting the rotatory inertia term in equation (5.2.1) and then substituting $\psi_r = -\frac{\partial w}{\partial r}$ in the resulting equation. The averaging shear constant is taken to be $\frac{\pi^2}{12}$.

Figures 5.1(a,b,c) show the convergence of the frequency parameter Ω with the number of collocation points for the first three modes of vibration for all the three edge conditions. In all

the computations $m = 15$ was fixed, since further increase in the value of m does not improve the results even in fourth place of decimal.

The results are given in Tables (5.1-5.9) and Figures (5.2-5.7). Tables (5.1-5.9) present the frequency parameter obtained by CPT (Ω_c) for $h_0 = 0.1$ and SPT (Ω_s) for $h_0 = 0.05, 0.1, 0.2$ taking various values of $\mu = -0.5, 0.0, 1.0$, $\eta = -0.5, 0.0, 1.0$, $\alpha = -0.5, 0.0, 0.5$ and $\beta = -0.5, 0.0, 0.5$ for clamped, simply supported and free plates respectively. In the case of classical theory, the frequencies for $h_0 = 0.1$ have been obtained by using a multiplying factor $h_0 / \sqrt{12}$. From the results, it has been found that for $\alpha > 0$, $\beta > 0$, the frequency parameter for free plate is smaller than that for clamped plate and higher than that for simply supported plate. The frequency parameter increases with the increase in non-homogeneity parameter μ, taper parameter α and β, thickness parameter h_0, while it decreases with the increase in density parameter η. From the results for Linear Thickness Variation (LTV), i.e. $\beta = 0.0$ and Parabolic Thickness Variation (PTV) i.e. $\alpha = 0.0$, it is noticed that when the plate becomes thicker towards the edge (i.e. $\alpha > 0$, $\beta > 0$), $\Omega_{LTV} > \Omega_{PTV}$, while it is just reverse when the plate becomes thicker towards the centre (i.e. $\alpha < 0, \beta < 0$).

Figures 5.2(a,b,c) show the effect of non-homogeneity parameter μ on frequency parameter Ω for all the three edge conditions for the first three modes of vibration for $\eta = -0.5$, $\alpha = 0.5$, $\beta = 0.5$ and $h_0 = 0.05, 0.1$ by CPT and SPT. It is observed that the frequency parameter Ω increases with increasing values of non-homogeneity parameter μ for all the three plates. The rate of increase of Ω with non-homogeneity parameter μ for clamped plate is higher as compared to that for simply supported and free plates. Further, it also increases with the increase in thickness h_0 as well as with increasing number of modes. The effect of transverse

shear and rotatory inertia increases with increasing value of μ. This effect also increases with increase in number of modes.

Figure 5.3a shows the plot of frequency parameter Ω versus density parameter η for $\mu = 1.0$, $\alpha = 0.5$, $\beta = 0.5$ and two values of $h_0 = 0.05, 0.1$ for clamped, simply supported and free plates vibrating in fundamental mode applying CPT and SPT. It is seen that frequency parameter Ω decreases with increasing values of density parameter η for all the three plates. The rate of decrease for simply supported plate is lower than that for clamped and free plates. The effect of transverse shear and rotatory inertia decreases with the increasing values of η. The difference between Ω_c and Ω_s is not appreciable for $h_0 = 0.05$ for simply supported and free edge conditions. However, when h_0 increases, this difference also increases. The discrepancy in Ω_c and Ω_s is larger for clamped plate as compared to those for free and simply supported plates. A similar inference is drawn when the plate vibrates in second and third mode (Figures 5.3(b,c)).

Figures 5.4(a,b,c) depict the variation of Ω with taper parameter α for $\mu = 1.0$, $\eta = -0.5$, $h_0 = 0.05, 0.1$ and $\beta = 0.5$ for all the three plates vibrating in fundamental, second and third mode respectively. It is observed that frequency parameter increases with increasing values of taper parameter α. The increase is more pronounced in the case of clamped plates. The effect of transverse shear and rotatory inertia is found to be more pronounced when α and β are both positive. Figures 5.5(a,b,c) show the effect of taper parameter β on first three frequency parameters Ω for $\mu = 1.0$, $\eta = -0.5$, $h_0 = 0.05, 0.1$ and $\alpha = 0.5$ for all the three edge conditions. It is clear that frequency parameter increases with increasing value of taper parameter β. The rate of increase is higher for clamped plate as compared to that for simply supported and free plates.

The rate of increase of Ω with α and β for second and third modes is much higher as compared to that for fundamental mode.

Figures 5.6(a,b,c) show the behaviour of frequency parameters Ω with thickness parameter h_0 for $\mu = 1.0$, $\eta = -0.5$, $\alpha = 0.5$ and $\beta = 0.5$ for first three modes of vibration for clamped, simply supported and free plate respectively. It is seen that the effect of transverse shear and rotatory inertia increases with the increase in the value of h_0 as well as the number of modes. This effect increases in the order of edge conditions, namely simply supported, free and clamped.

Figures 5.7(a,b,c) show the plots for normalized transverse displacements for $\mu = 1.0$, $\eta = -0.5$, $h_0 = 0.1$, $\alpha = 0.0$, $\beta = 0.0$; $\alpha = 0.5$, $\beta = 0.0$; $\alpha = 0.5$, $\beta = 0.5$ for the first three modes of vibration for clamped, simply supported and free plate respectively. The radii of nodal circles decrease as the outer edge becomes thicker and thicker for all three edge conditions except for the fundamental mode in the case of free plate. In this case the behaviour is just the reverse. The results show that the effect of transverse shear and rotatory inertia plays an important role in case of moderately thick non-homogeneous circular plates and hence can not be neglected for $h_0 > 0.1$ as reported by Deresiewicz and Mindlin[1955] in their significant contribution for circular disks.

A comparison of results for homogeneous ($\mu = 0.0$, $\eta = 0.0$) Mindlin's circular plates of uniform thickness ($\alpha = 0.0$, $\beta = 0.0$) with the exact results obtained by Irie et al.[1980] and DQM results obtained by Liew et al.[1997] for the first three natural frequencies, has been presented in table 5.10. It is seen that there is a close agreement of the results and that the Chebyshev collocation technique, used in this investigation is quite versatile.

Table 5.1

Values of frequency parameter Ω for clamped plate for η = -0.5

Mode	α	β	μ = -0.5					μ = 0.0					μ = 1.0				
			*	Ωc	Ωs			*	Ωc	Ωs			*	Ωc	Ωs		
				h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2
I	-0.5	0	5.7823	0.1669	0.0832	0.1648	0.3177	6.7376	0.1945	0.0969	0.1921	0.3711	9.1707	0.2647	0.1320	0.2617	0.5064
		0.5	8.9842	0.2594	0.1289	0.2533	0.4748	10.5316	0.3040	0.1511	0.2971	0.5577	14.4740	0.4178	0.2077	0.4083	0.7673
	0	-0.5	6.2569	0.1806	0.0900	0.1779	0.3407	7.2797	0.2101	0.1047	0.2071	0.3977	9.8979	0.2857	0.1424	0.2819	0.5427
		0	9.5005	0.2743	0.1362	0.2669	0.4962	11.1464	0.3218	0.1598	0.3134	0.5837	15.3791	0.4440	0.2205	0.4325	0.8064
		0.5	12.6893	0.3663	0.1812	0.3516	0.6333	14.9226	0.4308	0.2132	0.4138	0.7465	20.6014	0.5947	0.2943	0.5712	1.0306
	0.5	-0.5	9.9883	0.2883	0.1430	0.2796	0.5154	11.7224	0.3384	0.1679	0.3284	0.6069	16.2153	0.4681	0.2323	0.4545	0.8411
		0	13.2232	0.3817	0.1886	0.3647	0.6504	15.5730	0.4496	0.2222	0.4299	0.7681	21.6005	0.6236	0.3082	0.5961	1.0655
		0.5	16.4301	0.4743	0.2333	0.4459	0.7692	19.3602	0.5589	0.2750	0.5258	0.9089	26.7851	0.7732	0.3804	0.7271	1.2567
II	-0.5	0	26.9194	0.7771	0.3841	0.7436	1.3320	30.8689	0.8911	0.4405	0.8530	1.5298	40.4535	1.1678	0.5772	1.1175	2.0032
		0.5	36.0125	1.0396	0.5096	0.9653	1.6318	41.2360	1.1904	0.5838	1.1071	1.8774	53.8464	1.5544	0.7625	1.4471	2.4604
	0	-0.5	29.5624	0.8534	0.4207	0.8088	1.4211	33.9258	0.9794	0.4829	0.9287	1.6339	44.5205	1.2852	0.6336	1.2183	2.1428
		0	38.9153	1.1234	0.5488	1.0304	1.7066	44.5753	1.2868	0.6289	1.1825	1.9653	58.2294	1.6809	0.8218	1.5466	2.5781
		0.5	46.9682	1.3559	0.6560	1.2036	1.8971	53.7238	1.5509	0.7510	1.3813	2.1884	69.9704	2.0199	0.9790	1.8047	2.8758
	0.5	-0.5	41.7988	1.2066	0.5872	1.0927	1.7738	47.9016	1.3828	0.6734	1.2549	2.0449	62.6145	1.8075	0.8805	1.6427	2.6859
		0	50.0417	1.4446	0.6957	1.2637	1.9539	57.2557	1.6528	0.7969	1.4512	2.2559	74.5872	2.1531	1.0392	1.8973	2.9673
		0.5	57.5710	1.6619	0.7918	1.4049	2.0852	65.7867	1.8991	0.9062	1.6138	2.4107	85.4815	2.4676	1.1796	2.1092	3.1761
III	-0.5	0	62.6436	1.8084	0.8825	1.6538	2.7321	71.6301	2.0678	1.0091	1.8908	3.1234	93.0924	2.6873	1.3104	2.4510	4.0347
		0.5	81.0526	2.3398	1.1238	2.0303	3.1173	92.2944	2.6643	1.2805	2.3170	3.5678	118.9703	3.4344	1.6510	2.9897	4.6123
	0	-0.5	69.3736	2.0026	0.9721	1.7985	2.8920	79.3070	2.2894	1.1112	2.0556	3.3050	102.9940	2.9732	1.4417	2.6616	4.2640
		0	88.1704	2.5453	1.2140	2.1615	3.2388	100.3867	2.8979	1.3832	2.4669	3.7062	129.3342	3.7336	1.7825	3.1815	4.7879
		0.5	103.8901	2.9990	1.4053	2.4170	3.4422	117.9711	3.4035	1.5985	2.7587	3.9440	151.2091	4.3650	2.0524	3.5549	5.1057
	0.5	-0.5	95.2048	2.7483	1.3012	2.2834	3.3456	108.3867	3.1289	1.4826	2.6062	3.8275	139.5838	4.0294	1.9098	3.3598	4.9413
		0	111.2158	3.2105	1.4916	2.5271	3.5302	126.2833	3.6455	1.6968	2.8849	4.0435	161.8083	4.6710	2.1781	3.7171	5.2310
		0.5	125.5564	3.6245	1.6521	2.7122	3.6585	142.2857	4.1074	1.8779	3.0982	4.1930	181.6230	5.2430	2.4057	3.9942	5.4316

* for general value of h_0

Table 5.2

Values of frequency parameter Ω for clamped plate for η = 0.0

Mode	α	β	μ = -0.5					μ = 0.0					μ = 1.0				
			*	Ωc	Ωs			*	Ωc	Ωs			*	Ωc	Ωs		
				h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2
I	-0.5	0	5.2676	0.1521	0.0758	0.1501	0.2891	6.1504	0.1775	0.0885	0.1753	0.3384	8.4058	0.2427	0.1210	0.2398	0.4639
		0.5	8.2329	0.2377	0.1181	0.2319	0.4339	9.6732	0.2792	0.1388	0.2727	0.5111	13.3572	0.3856	0.1916	0.3767	0.7069
	0	-0.5	5.6889	0.1642	0.0818	0.1617	0.3094	6.6320	0.1914	0.0954	0.1886	0.3619	9.0529	0.2613	0.1302	0.2578	0.4959
		0	8.6879	0.2508	0.1245	0.2438	0.4523	10.2158	0.2949	0.1464	0.2870	0.5335	14.1597	0.4088	0.2030	0.3980	0.7411
		0.5	11.6473	0.3362	0.1663	0.3222	0.5782	13.7310	0.3964	0.1961	0.3802	0.6838	19.0532	0.5500	0.2721	0.5277	0.9500
	0.5	-0.5	9.1180	0.2632	0.1305	0.2550	0.4689	10.7241	0.3096	0.1536	0.3002	0.5535	14.8998	0.4301	0.2134	0.4174	0.7712
		0	12.1153	0.3497	0.1727	0.3335	0.5926	14.3021	0.4129	0.2040	0.3941	0.7019	19.9359	0.5755	0.2843	0.5496	0.9798
		0.5	15.0951	0.4358	0.2142	0.4085	0.7011	17.8331	0.5148	0.2531	0.4831	0.8313	24.8054	0.7161	0.3521	0.6721	1.1578
II	-0.5	0	23.7356	0.6852	0.3386	0.6553	1.1725	27.3002	0.7881	0.3896	0.7543	1.3520	35.9954	1.0391	0.5137	0.9947	1.7846
		0.5	32.0322	0.9247	0.4530	0.8568	1.4424	36.7861	1.0619	0.5206	0.9860	1.6669	48.3178	1.3948	0.6841	1.2978	2.2038
	0	-0.5	26.0749	0.7527	0.3710	0.7129	1.2508	30.0152	0.8665	0.4272	0.8214	1.4441	39.6350	1.1442	0.5642	1.0850	1.9099
		0	34.6170	0.9993	0.4878	0.9143	1.5077	39.7711	1.1481	0.5608	1.0530	1.7442	52.2669	1.5088	0.7375	1.3873	2.3093
		0.5	42.0016	1.2125	0.5858	1.0715	1.6775	48.1833	1.3909	0.6728	1.2343	1.9446	63.1185	1.8221	0.8826	1.6247	2.5806
	0.5	-0.5	37.1793	1.0733	0.5219	0.9692	1.5660	42.7391	1.2338	0.6004	1.1172	1.8139	56.2127	1.6227	0.7903	1.4736	2.4056
		0	44.7438	1.2916	0.6211	1.1243	1.7264	51.3480	1.4823	0.7138	1.2962	2.0034	67.2899	1.9425	0.9369	1.7078	2.6622
		0.5	51.6715	1.4916	0.7090	1.2519	1.8426	59.2181	1.7095	0.8142	1.4440	2.1415	77.3894	2.2340	1.0667	1.9024	2.8512
III	-0.5	0	54.9910	1.5875	0.7747	1.4514	2.3963	63.0611	1.8204	0.8885	1.6651	2.7518	82.4331	2.3796	1.1609	2.1739	3.5872
		0.5	71.7238	2.0705	0.9935	1.7908	2.7387	81.8995	2.3642	1.1355	2.0515	3.1498	106.1598	3.0646	1.4732	2.6671	4.1123
	0	-0.5	60.9310	1.7589	0.8537	1.5790	2.5376	69.8625	2.0168	0.9789	1.8114	2.9136	91.2716	2.6348	1.2783	2.3630	3.7948
		0	78.0372	2.2527	1.0732	1.9063	2.8457	89.1041	2.5722	1.2268	2.1842	3.2726	115.4552	3.3329	1.5911	2.8393	4.2706
		0.5	92.3972	2.6673	1.2469	2.1351	3.0250	105.2133	3.0372	1.4230	2.4472	3.4832	135.6044	3.9146	1.8390	3.1799	4.5561
	0.5	-0.5	84.2756	2.4328	1.1504	2.0136	2.9398	96.2259	2.7778	1.3151	2.3077	3.3803	124.6483	3.5983	1.7053	2.9994	4.4090
		0	98.9144	2.8554	1.3232	2.2316	3.1027	112.6360	3.2515	1.5104	2.5586	3.5718	145.1395	4.1898	1.9520	3.3252	4.6691
		0.5	112.0582	3.2348	1.4687	2.3959	3.2164	127.3436	3.6761	1.6753	2.7494	3.7045	163.4471	4.7183	2.1610	3.5765	4.8486

* for general value of h_0

Table 5.3

Values of frequency parameter Ω for clamped plate for $\eta = 1.0$

Mode	α	β	$\mu = -0.5$					$\mu = 0.0$					$\mu = 1.0$				
			*	Ωc	Ωs			*	Ωc	Ωs			*	Ωc	Ωs		
				h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2
I	-0.5	0	4.3141	0.1245	0.0620	0.1228	0.2361	5.0589	0.1460	0.0728	0.1441	0.2777	6.9743	0.2013	0.1004	0.1989	0.3842
		0.5	6.8269	0.1971	0.0979	0.1919	0.3573	8.0606	0.2327	0.1156	0.2268	0.4233	11.2424	0.3245	0.1612	0.3166	0.5923
	0	-0.5	4.6407	0.1340	0.0667	0.1317	0.2514	5.4328	0.1568	0.0781	0.1544	0.2955	7.4779	0.2159	0.1075	0.2128	0.4088
		0	7.1733	0.2071	0.1027	0.2009	0.3705	8.4746	0.2446	0.1214	0.2376	0.4395	11.8597	0.3424	0.1699	0.3328	0.6175
		0.5	9.6912	0.2798	0.1382	0.2670	0.4749	11.4843	0.3315	0.1638	0.3168	0.5652	16.1078	0.4650	0.2298	0.4448	0.7960
	0.5	-0.5	7.5016	0.2166	0.1073	0.2092	0.3824	8.8627	0.2558	0.1268	0.2475	0.4539	12.4275	0.3588	0.1779	0.3475	0.6395
		0	10.0431	0.2899	0.1430	0.2753	0.4843	11.9150	0.3440	0.1697	0.3270	0.5774	16.7809	0.4844	0.2391	0.4611	0.8168
		0.5	12.5840	0.3633	0.1782	0.3383	0.5730	14.9471	0.4315	0.2118	0.4025	0.6843	21.0272	0.6070	0.2981	0.5670	0.9678
II	-0.5	0	18.3158	0.5287	0.2612	0.5047	0.8993	21.1908	0.6117	0.3023	0.5847	1.0449	28.2756	0.8162	0.4035	0.7813	1.4012
		0.5	25.1528	0.7261	0.3551	0.6688	1.1136	29.0528	0.8387	0.4106	0.7749	1.2982	38.6035	1.1144	0.5462	1.0341	1.7468
	0	-0.5	20.1283	0.5811	0.2862	0.5490	0.9585	23.3085	0.6729	0.3316	0.6368	1.1155	31.1577	0.8994	0.4435	0.8528	1.4999
		0	27.1791	0.7846	0.3823	0.7130	1.1620	31.4115	0.9068	0.4422	0.8271	1.3565	41.7754	1.2060	0.5890	1.1054	1.8295
		0.5	33.3287	0.9621	0.4632	0.8403	1.2938	38.4559	1.1101	0.5354	0.9753	1.5145	50.9585	1.4710	0.7112	1.3033	2.0497
	0.5	-0.5	29.1787	0.8423	0.4087	0.7548	1.2046	33.7466	0.9742	0.4732	0.8766	1.4084	44.9341	1.2971	0.6311	1.1737	1.9043
		0	35.4855	1.0244	0.4907	0.8802	1.3288	40.9669	1.1826	0.5676	1.0229	1.5575	54.3272	1.5683	0.7548	1.3691	2.1125
		0.5	41.2949	1.1921	0.5633	0.9824	1.4178	47.6025	1.3742	0.6512	1.1427	1.6649	62.9284	1.8166	0.8644	1.5303	2.2639
III	-0.5	0	42.0591	1.2141	0.5921	1.1074	1.8211	48.5050	1.4002	0.6832	1.2793	2.1095	64.1358	1.8514	0.9036	1.6936	2.7997
		0.5	55.7456	1.6092	0.7699	1.3787	2.0856	64.0059	1.8477	0.8854	1.5913	2.4212	83.8865	2.4216	1.1629	2.1003	3.2232
	0	-0.5	46.6404	1.3464	0.6529	1.2053	1.9291	53.7904	1.5528	0.7534	1.3927	2.2351	71.1094	2.0528	0.9964	1.8437	2.9659
		0	60.6645	1.7512	0.8315	1.4666	2.1668	69.6613	2.0109	0.9566	1.6937	2.5158	91.2929	2.6354	1.2567	2.2366	3.3489
		0.5	72.5320	2.0938	0.9727	1.6463	2.3040	83.0515	2.3975	1.1175	1.9029	2.6782	108.2282	3.1243	1.4632	2.5145	3.5740
	0.5	-0.5	65.5218	1.8915	0.8911	1.5482	2.2385	75.2493	2.1723	1.0255	1.7888	2.5990	98.6172	2.8468	1.3475	2.3633	3.4591
		0	77.6391	2.2412	1.0315	1.7190	2.3635	88.9129	2.5667	1.1855	1.9880	2.7469	115.8719	3.3449	1.5531	2.6289	3.6640
		0.5	88.5764	2.5570	1.1491	1.8459	2.4525	101.2208	2.9220	1.3201	2.1372	2.8508	131.3580	3.7920	1.7269	2.8310	3.8055

* for general value of h_0

Table 5.4

Values of frequency parameter Ω for simply supported plate for $\eta = -0.5$

Mode	α	β	μ = -0.5					μ = 0.0					μ = 1.0				
			*	Ωc	Ωs			*	Ωc	Ωs			*	Ωc	Ωs		
				h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2
I	-0.5	0	3.4712	0.1002	0.0500	0.0997	0.1965	3.9408	0.1138	0.0568	0.1132	0.2234	5.0531	0.1459	0.0729	0.1453	0.2873
		0.5	4.3703	0.1262	0.0630	0.1253	0.2454	4.9682	0.1434	0.0716	0.1425	0.2795	6.4540	0.1863	0.0930	0.1852	0.3641
	0	-0.5	3.9578	0.1143	0.0570	0.1135	0.2227	4.4870	0.1295	0.0647	0.1288	0.2531	5.7368	0.1656	0.0827	0.1648	0.3248
		0	4.8362	0.1396	0.0697	0.1384	0.2699	5.4854	0.1583	0.0790	0.1571	0.3069	7.0874	0.2046	0.1021	0.2031	0.3980
		0.5	5.7008	0.1646	0.0820	0.1626	0.3144	6.5136	0.1880	0.0937	0.1859	0.3602	8.5969	0.2482	0.1238	0.2456	0.4770
	0.5	-0.5	5.3019	0.1531	0.0763	0.1514	0.2939	6.0043	0.1733	0.0865	0.1717	0.3339	7.7289	0.2231	0.1113	0.2212	0.4319
		0	6.1453	0.1774	0.0884	0.1750	0.3367	7.0035	0.2022	0.1008	0.1996	0.3850	9.1877	0.2652	0.1322	0.2621	0.5072
		0.5	7.0302	0.2029	0.1010	0.1993	0.3793	8.0765	0.2331	0.1161	0.2292	0.4372	10.8085	0.3120	0.1554	0.3070	0.5873
II	-0.5	0	21.1646	0.6110	0.3033	0.5942	1.1045	24.1560	0.6973	0.3462	0.6781	1.2604	31.3380	0.9047	0.4490	0.8789	1.6296
		0.5	26.5982	0.7678	0.3797	0.7357	1.3242	30.2135	0.8722	0.4313	0.8359	1.5052	38.7592	1.1189	0.5532	1.0715	1.9258
	0	-0.5	23.7449	0.6855	0.3397	0.6620	1.2112	27.1551	0.7839	0.3884	0.7570	1.3847	35.3799	1.0213	0.5059	0.9849	1.7967
		0	29.4088	0.8490	0.4188	0.8066	1.4275	33.4691	0.9662	0.4767	0.9182	1.6257	43.0951	1.2440	0.6136	1.1810	2.0862
		0.5	34.2642	0.9891	0.4857	0.9246	1.5884	38.8520	1.1216	0.5509	1.0491	1.8039	49.6169	1.4323	0.7034	1.3388	2.2982
	0.5	-0.5	32.1785	0.9289	0.4571	0.8746	1.5213	36.6866	1.0591	0.5212	0.9973	1.7358	47.4039	1.3684	0.6731	1.2869	2.2340
		0	37.1930	1.0737	0.5257	0.9934	1.6776	42.2449	1.2195	0.5972	1.1291	1.9087	54.1239	1.5624	0.7649	1.4452	2.4381
		0.5	41.7869	1.2063	0.5876	1.0969	1.8033	47.3208	1.3660	0.6657	1.2436	2.0471	60.2327	1.7388	0.8472	1.5822	2.6002
III	-0.5	0	53.3298	1.5395	0.7562	1.4404	2.4772	60.8744	1.7573	0.8629	1.6425	2.8198	78.8013	2.2748	1.1158	2.1182	3.6133
		0.5	67.0534	1.9357	0.9416	1.7519	2.8586	76.0816	2.1963	1.0684	1.9877	3.2431	97.2716	2.8080	1.3649	2.5349	4.1205
	0	-0.5	59.7779	1.7256	0.8439	1.5901	2.6659	68.2611	1.9705	0.9633	1.8136	3.0348	88.4161	2.5524	1.2461	2.3386	3.8866
		0	73.7511	2.1290	1.0301	1.8932	3.0149	83.7329	2.4172	1.1694	2.1492	3.4223	107.1722	3.0938	1.4954	2.7428	4.3505
		0.5	85.5072	2.4684	1.1817	2.1237	3.2487	96.7041	2.7916	1.3370	2.4045	3.6823	122.7872	3.5446	1.6969	3.0494	4.6613
	0.5	-0.5	80.3894	2.3206	1.1163	2.0262	3.1528	91.3164	2.6361	1.2680	2.3013	3.5805	116.9831	3.3770	1.6226	2.9383	4.5536
		0	92.3434	2.6657	1.2678	2.2488	3.3671	104.4969	3.0166	1.4353	2.5477	3.8189	132.8232	3.8343	1.8235	3.2337	4.8383
		0.5	103.0955	2.9761	1.3996	2.4306	3.5255	116.3255	3.3580	1.5806	2.7491	3.9947	146.9777	4.2429	1.9975	3.4752	5.0468

* for general value of h_0

Table 5.5

Values of frequency parameter Ω for simply supported plate for $\eta = 0.0$

Mode	α	β	μ = -0.5					μ = 0.0					μ = 1.0				
			*	Ωc	Ωs			*	Ωc	Ωs			*	Ωc	Ωs		
				h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2
I	-0.5	0	3.1222	0.0901	0.0450	0.0897	0.1767	3.5498	0.1025	0.0512	0.1020	0.2012	4.5644	0.1318	0.0658	0.1312	0.2595
		0.5	3.9284	0.1134	0.0566	0.1126	0.2203	4.4716	0.1291	0.0644	0.1282	0.2513	5.8231	0.1681	0.0839	0.1671	0.3282
	0	-0.5	3.5575	0.1027	0.0513	0.1020	0.2000	4.0392	0.1166	0.0582	0.1159	0.2277	5.1786	0.1495	0.0747	0.1487	0.2931
		0	4.3455	0.1254	0.0626	0.1243	0.2422	4.9351	0.1425	0.0711	0.1413	0.2758	6.3917	0.1845	0.0921	0.1832	0.3586
		0.5	5.1172	0.1477	0.0736	0.1459	0.2815	5.8537	0.1690	0.0842	0.1670	0.3229	7.7432	0.2235	0.1115	0.2211	0.4288
	0.5	-0.5	4.7624	0.1375	0.0685	0.1360	0.2636	5.4002	0.1559	0.0777	0.1543	0.3000	6.9676	0.2011	0.1003	0.1994	0.3890
		0	5.5152	0.1592	0.0793	0.1569	0.3013	6.2927	0.1817	0.0905	0.1792	0.3450	8.2732	0.2388	0.1190	0.2359	0.4558
		0.5	6.3033	0.1820	0.0905	0.1784	0.3386	7.2492	0.2093	0.1041	0.2054	0.3908	9.7218	0.2806	0.1397	0.2758	0.5263
II	-0.5	0	18.5522	0.5356	0.2659	0.5210	0.9690	21.2386	0.6131	0.3044	0.5965	1.1097	27.7240	0.8003	0.3973	0.7781	1.4456
		0.5	23.5495	0.6798	0.3361	0.6511	1.1709	26.8323	0.7746	0.3830	0.7423	1.3362	34.6339	0.9998	0.4943	0.9578	1.7231
	0	-0.5	20.8236	0.6011	0.2979	0.5807	1.0630	23.8870	0.6896	0.3417	0.6662	1.2199	31.3192	0.9041	0.4479	0.8727	1.5954
		0	26.0325	0.7515	0.3707	0.7137	1.2619	29.7200	0.8579	0.4232	0.8152	1.4431	38.5132	1.1118	0.5484	1.0559	1.8674
		0.5	30.5313	0.8814	0.4326	0.8227	1.4103	34.7290	1.0025	0.4923	0.9369	1.6087	44.6324	1.2884	0.6327	1.2042	2.0671
	0.5	-0.5	28.4755	0.8220	0.4044	0.7735	1.3443	32.5691	0.9402	0.4626	0.8852	1.5404	42.3620	1.2229	0.6016	1.1507	2.0001
		0	33.1209	0.9561	0.4679	0.8833	1.4884	37.7423	1.0895	0.5334	1.0078	1.7012	48.6734	1.4051	0.6879	1.2996	2.1928
		0.5	37.3972	1.0796	0.5255	0.9792	1.6045	42.4881	1.2265	0.5973	1.1144	1.8300	54.4319	1.5713	0.7654	1.4287	2.3458
III	-0.5	0	46.6827	1.3476	0.6621	1.2619	2.1734	53.4404	1.5427	0.7578	1.4437	2.4844	69.5828	2.0087	0.9859	1.8746	3.2104
		0.5	59.2161	1.7094	0.8314	1.5462	2.5211	67.3808	1.9451	0.9462	1.7605	2.8730	86.6428	2.5012	1.2163	2.2611	3.6833
	0	-0.5	52.3469	1.5111	0.7392	1.3938	2.3403	59.9533	1.7307	0.8465	1.5952	2.6758	78.1234	2.2552	1.1019	2.0717	3.4574
		0	65.1256	1.8800	0.9094	1.6707	2.6585	74.1561	2.1407	1.0357	1.9036	3.0319	95.4730	2.7561	1.3329	2.4473	3.8904
		0.5	75.9212	2.1917	1.0484	1.8812	2.8708	86.1120	2.4858	1.1899	2.1381	3.2698	109.9741	3.1747	1.5201	2.7324	4.1795
	0.5	-0.5	70.9833	2.0491	0.9855	1.7879	2.7798	80.8729	2.3346	1.1230	2.0383	3.1722	104.2267	3.0088	1.4465	2.6226	4.0735
		0	81.9714	2.3663	1.1244	1.9911	2.9743	93.0342	2.6857	1.2772	2.2648	3.3904	118.9535	3.4339	1.6333	2.8975	4.3384
		0.5	91.8815	2.6524	1.2454	2.1569	3.1173	103.9770	3.0016	1.4112	2.4497	3.5505	132.1455	3.8147	1.7954	3.1222	4.5313

* for general value of h_0

Table 5.6

Values of frequency parameter Ω for simply supported plate for $\eta = 1.0$

Mode	α	β	$\mu = -0.5$					$\mu = 0.0$					$\mu = 1.0$				
			*	Ωc	Ωs			*	Ωc	Ωs			*	Ωc	Ωs		
				h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2
I	-0.5	0	2.4868	0.0718	0.0358	0.0714	0.1406	2.8360	0.0819	0.0409	0.0815	0.1606	3.6678	0.1059	0.0529	0.1055	0.2084
		0.5	3.1239	0.0902	0.0450	0.0894	0.1746	3.5653	0.1029	0.0514	0.1021	0.1998	4.6660	0.1347	0.0672	0.1338	0.2624
	0	-0.5	2.8303	0.0817	0.0408	0.0811	0.1590	3.2236	0.0931	0.0465	0.0925	0.1815	4.1571	0.1200	0.0599	0.1194	0.2351
		0	3.4540	0.0997	0.0497	0.0987	0.1918	3.9330	0.1135	0.0566	0.1125	0.2191	5.1187	0.1478	0.0737	0.1466	0.2865
		0.5	4.0582	0.1172	0.0584	0.1154	0.2217	4.6532	0.1343	0.0669	0.1325	0.2550	6.1828	0.1785	0.0890	0.1763	0.3404
	0.5	-0.5	3.7838	0.1092	0.0544	0.1079	0.2087	4.3019	0.1242	0.0619	0.1228	0.2381	5.5774	0.1610	0.0803	0.1595	0.3105
		0	4.3736	0.1263	0.0629	0.1242	0.2372	5.0018	0.1444	0.0719	0.1422	0.2724	6.6047	0.1907	0.0950	0.1880	0.3617
		0.5	4.9877	0.1440	0.0716	0.1407	0.2648	5.7486	0.1659	0.0825	0.1624	0.3064	7.7411	0.2235	0.1112	0.2190	0.4149
II	-0.5	0	14.1696	0.4090	0.2031	0.3979	0.7398	16.3177	0.4711	0.2339	0.4584	0.8533	21.5609	0.6224	0.3090	0.6057	1.1279
		0.5	18.3632	0.5301	0.2620	0.5068	0.9083	21.0515	0.6077	0.3004	0.5816	1.0447	27.5109	0.7942	0.3927	0.7608	1.3686
	0	-0.5	15.9085	0.4592	0.2276	0.4435	0.8116	18.3578	0.5299	0.2626	0.5121	0.9383	24.3697	0.7035	0.3486	0.6799	1.2461
		0	20.2762	0.5853	0.2885	0.5547	0.9772	23.2937	0.6724	0.3316	0.6380	1.1267	30.5755	0.8826	0.4354	0.8383	1.4828
		0.5	24.1079	0.6959	0.3412	0.6469	1.1016	27.5988	0.7967	0.3908	0.7420	1.2677	35.9295	1.0372	0.5091	0.9679	1.6578
	0.5	-0.5	22.1519	0.6395	0.3144	0.6003	1.0391	25.4982	0.7361	0.3620	0.6920	1.2009	33.6053	0.9701	0.4773	0.9129	1.5872
		0	26.1046	0.7536	0.3683	0.6929	1.1596	29.9426	0.8644	0.4227	0.7966	1.3376	39.1333	1.1297	0.5528	1.0432	1.7565
		0.5	29.7804	0.8597	0.4175	0.7742	1.2572	34.0603	0.9832	0.4780	0.8882	1.4477	44.2205	1.2765	0.6212	1.1567	1.8912
III	-0.5	0	35.5298	1.0257	0.5040	0.9607	1.6556	40.9040	1.1808	0.5802	1.1063	1.9078	53.8779	1.5553	0.7641	1.4559	2.5067
		0.5	45.8806	1.3245	0.6435	1.1941	1.9386	52.5035	1.5156	0.7368	1.3691	2.2289	68.2906	1.9714	0.9588	1.7835	2.9092
	0	-0.5	39.8575	1.1506	0.5629	1.0615	1.7833	45.9172	1.3255	0.6485	1.2233	2.0565	60.5504	1.7479	0.8549	1.6113	2.7042
		0	50.4382	1.4560	0.7035	1.2892	2.0425	57.7665	1.6676	0.8062	1.4797	2.3509	75.2496	2.1723	1.0508	1.9306	3.0735
		0.5	59.4550	1.7163	0.8189	1.4618	2.2134	67.8279	1.9580	0.9354	1.6744	2.5457	87.6393	2.5299	1.2105	2.1732	3.3182
	0.5	-0.5	54.9546	1.5864	0.7619	1.3786	2.1340	62.9843	1.8182	0.8739	1.5837	2.4586	82.1519	2.3715	1.1405	2.0692	3.2188
		0	64.1490	1.8518	0.8774	1.5454	2.2905	73.2383	2.1142	1.0032	1.7719	2.6372	94.7611	2.7355	1.3002	2.3036	3.4432
		0.5	72.4891	2.0926	0.9780	1.6807	2.4043	82.5168	2.3821	1.1158	1.9250	2.7666	106.1165	3.0633	1.4391	2.4946	3.6044

* for general value of h_0

Table 5.7

Values of frequency parameter Ω for free plate for η = -0.5

Mode	α	β	μ = -0.5					μ = 0.0					μ = 1.0				
			*	Ωc	Ωs			*	Ωc	Ωs			*	Ωc	Ωs		
				h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2
I	-0.5	0	7.6512	0.2209	0.1102	0.2189	0.4271	8.5256	0.2461	0.1228	0.2441	0.4765	10.5669	0.3050	0.1522	0.3027	0.5918
		0.5	8.2370	0.2378	0.1185	0.2348	0.4530	9.2473	0.2669	0.1331	0.2637	0.5095	11.7358	0.3388	0.1689	0.3348	0.6480
		-0.5	8.8436	0.2553	0.1273	0.2524	0.4889	9.8303	0.2838	0.1415	0.2807	0.5444	12.1335	0.3503	0.1747	0.3467	0.6737
	0	0	9.2861	0.2681	0.1335	0.2639	0.5058	10.3962	0.3001	0.1495	0.2957	0.5675	13.1162	0.3786	0.1886	0.3733	0.7178
		0.5	10.1677	0.2935	0.1460	0.2873	0.5423	11.4956	0.3318	0.1650	0.3250	0.6145	14.8727	0.4293	0.2136	0.4208	0.7969
		-0.5	10.3456	0.2987	0.1486	0.2932	0.5577	11.5593	0.3337	0.1661	0.3278	0.6246	14.5223	0.4192	0.2087	0.4122	0.7871
	0.5	0	11.1358	0.3215	0.1597	0.3136	0.5877	12.5551	0.3624	0.1801	0.3539	0.6643	16.1444	0.4660	0.2316	0.4554	0.8566
		0.5	12.1778	0.3515	0.1743	0.3402	0.6260	13.8588	0.4001	0.1984	0.3875	0.7143	18.2126	0.5258	0.2608	0.5096	0.9410
II	-0.5	0	28.9596	0.8360	0.4141	0.8066	1.4734	32.9017	0.9498	0.4705	0.9160	1.6716	42.1928	1.2180	0.6030	1.1726	2.1310
		0.5	34.3938	0.9929	0.4894	0.9404	1.6538	38.8313	1.1210	0.5525	1.0618	1.8674	49.1903	1.4200	0.6997	1.3436	2.3572
		-0.5	33.2679	0.9604	0.4746	0.9182	1.6461	37.8344	1.0922	0.5396	1.0436	1.8683	48.5867	1.4026	0.6925	1.3369	2.3808
	0	0	38.5853	1.1139	0.5474	1.0442	1.8030	43.6084	1.2589	0.6187	1.1801	2.0369	55.3258	1.5971	0.7846	1.4948	2.5711
		0.5	43.5454	1.2570	0.6141	1.1544	1.9239	49.0296	1.4154	0.6916	1.3006	2.1686	61.7573	1.7828	0.8710	1.6372	2.7249
		-0.5	42.8138	1.2359	0.6055	1.1460	1.9425	48.4339	1.3982	0.6849	1.2961	2.1956	61.5353	1.7764	0.8697	1.6430	2.7709
	0.5	0	47.7623	1.3788	0.6712	1.2513	2.0491	53.8273	1.5539	0.7566	1.4109	2.3108	67.8908	1.9598	0.9540	1.7774	2.9030
		0.5	52.4791	1.5149	0.7326	1.3449	2.1312	58.9816	1.7027	0.8237	1.5134	2.4002	74.0091	2.1365	1.0337	1.8990	3.0081
III	-0.5	0	64.7299	1.8686	0.9149	1.7291	2.9203	73.7404	2.1287	1.0419	1.9670	3.3142	94.9875	2.7421	1.3402	2.5218	4.2163
		0.5	79.1946	2.2862	1.1068	2.0373	3.2502	89.5657	2.5855	1.2517	2.3036	3.6731	113.7101	3.2825	1.5877	2.9159	4.6299
		-0.5	73.4333	2.1198	1.0326	1.9274	3.1685	83.6967	2.4161	1.1763	2.1930	3.5955	107.8786	3.1142	1.5135	2.8104	4.5690
	0	0	87.8164	2.5350	1.2198	2.2153	3.4480	99.3786	2.8688	1.3802	2.5059	3.8978	126.2877	3.6456	1.7519	3.1729	4.9119
		0.5	100.2487	2.8939	1.3754	2.4357	3.6191	112.9479	3.2605	1.5502	2.7470	4.0859	142.2783	4.1072	1.9520	3.4555	5.1323
		-0.5	96.4352	2.7838	1.3307	2.3840	3.6253	109.1903	3.1521	1.5064	2.6977	4.0989	138.8659	4.0087	1.9131	3.4161	5.1638
	0.5	0	108.9059	3.1438	1.4832	2.5902	3.7671	122.7715	3.5441	1.6726	2.9226	4.2547	154.7914	4.4684	2.1076	3.6779	5.3447
		0.5	120.2682	3.4718	1.6165	2.7565	3.8406	135.1474	3.9014	1.8182	3.1047	4.3405	169.3227	4.8879	2.2787	3.8920	5.4532

* for general value of h_0

Table 5.8

Values of frequency parameter Ω for free plate for $\eta = 0.0$

Mode	α	β	μ = -0.5					μ = 0.0					μ = 1.0				
			*	Ωc	Ωs			*	Ωc	Ωs			*	Ωc	Ωs		
				h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2
I	-0.5	0	6.5544	0.1892	0.0944	0.1876	0.3660	7.3210	0.2113	0.1054	0.2096	0.4094	9.1148	0.2631	0.1313	0.2611	0.5109
		0.5	7.1209	0.2056	0.1024	0.2029	0.3912	8.0104	0.2312	0.1153	0.2283	0.4409	10.2059	0.2946	0.1469	0.2911	0.5632
	0	-0.5	7.5871	0.2190	0.1092	0.2166	0.4197	8.4538	0.2440	0.1217	0.2414	0.4685	10.4805	0.3025	0.1509	0.2996	0.5825
		0	8.0258	0.2317	0.1154	0.2281	0.4367	9.0031	0.2599	0.1295	0.2560	0.4910	11.4019	0.3291	0.1640	0.3245	0.6238
		0.5	8.8302	0.2549	0.1267	0.2492	0.4694	10.0019	0.2887	0.1436	0.2825	0.5330	12.9879	0.3749	0.1865	0.3672	0.6943
	0.5	-0.5	8.9408	0.2581	0.1284	0.2533	0.4815	10.0093	0.2889	0.1438	0.2838	0.5405	12.6218	0.3644	0.1814	0.3582	0.6840
		0	9.6649	0.2790	0.1386	0.2719	0.5084	10.9165	0.3151	0.1566	0.3074	0.5759	14.0871	0.4067	0.2021	0.3971	0.7459
		0.5	10.6010	0.3060	0.1516	0.2956	0.5415	12.0854	0.3489	0.1729	0.3373	0.6192	15.9387	0.4601	0.2281	0.4453	0.8195
II	-0.5	0	25.1820	0.7269	0.3602	0.7018	1.2836	28.6948	0.8283	0.4104	0.7995	1.4617	37.0232	1.0688	0.5293	1.0303	1.8779
		0.5	30.2380	0.8729	0.4302	0.8264	1.4523	34.2393	0.9884	0.4872	0.9362	1.6462	43.6261	1.2594	0.6207	1.1923	2.0943
	0	-0.5	28.9266	0.8350	0.4127	0.7990	1.4346	33.0007	0.9526	0.4708	0.9112	1.6350	42.6575	1.2314	0.6083	1.1757	2.1012
		0	33.9100	0.9789	0.4810	0.9174	1.5832	38.4432	1.1098	0.5454	1.0404	1.7962	49.0768	1.4167	0.6961	1.3271	2.2864
		0.5	38.5366	1.1125	0.5432	1.0200	1.6956	43.5210	1.2563	0.6137	1.1532	1.9195	55.1422	1.5918	0.7777	1.4617	2.4319
	0.5	-0.5	37.6092	1.0857	0.5318	1.0064	1.7055	42.6843	1.2322	0.6037	1.1425	1.9364	54.5883	1.5758	0.7717	1.4592	2.4660
		0	42.2438	1.2195	0.5934	1.1049	1.8055	47.7593	1.3787	0.6711	1.2505	2.0452	60.6154	1.7498	0.8518	1.5871	2.5923
		0.5	46.6550	1.3468	0.6507	1.1920	1.8813	52.5980	1.5184	0.7340	1.3464	2.1283	66.3917	1.9166	0.9270	1.7016	2.6906
III	-0.5	0	56.4716	1.6302	0.7985	1.5103	2.5558	64.5147	1.8624	0.9120	1.7239	2.9127	83.5836	2.4129	1.1803	2.2254	3.7377
		0.5	69.7484	2.0135	0.9746	1.7933	2.8584	79.1053	2.2836	1.1056	2.0349	3.2451	100.9993	2.9156	1.4110	2.5942	4.1276
	0	-0.5	64.0613	1.8493	0.9012	1.6838	2.7740	73.2291	2.1139	1.0298	1.9226	3.1619	94.9521	2.7410	1.3335	2.4820	4.0548
		0	77.3162	2.2319	1.0737	1.9494	3.0318	87.7502	2.5331	1.2188	2.2134	3.4436	112.1630	3.2379	1.5570	2.8236	4.3810
		0.5	88.7758	2.5627	1.2169	2.1515	3.1845	100.3099	2.3957	1.3760	2.4358	3.6138	127.0838	3.6686	1.7438	3.0879	4.5848
	0.5	-0.5	84.8794	2.4503	1.1710	2.0974	3.1871	96.3928	2.7826	1.3300	2.3826	3.6214	123.3271	3.5601	1.7003	3.0408	4.6075
		0	96.4035	2.7829	1.3117	2.2869	3.3129	108.9980	3.1465	1.4842	2.5908	3.7618	138.2345	3.9905	1.8826	3.2866	4.7749
		0.5	106.9103	3.0862	1.4344	2.4386	3.3694	120.4877	3.4782	1.6189	2.7583	3.8316	151.8286	4.3829	2.0425	3.4862	4.8692

* for general value of h_0

Table 5.9

Values of frequency parameter Ω for free plate for η = 1.0

Mode	α	β	μ = -0.5					μ = 0.0					μ = 1.0				
			*	Ωc	Ωs			*	Ωc	Ωs			*	Ωc	Ωs		
				h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2		h_0=0.1	h_0=0.05	h_0=0.1	h_0=0.2
I	-0.5	0	4.8080	0.1388	0.0692	0.1376	0.2684	5.3956	0.1558	0.0777	0.1545	0.3018	6.7773	0.1956	0.0976	0.1942	0.3801
		0.5	5.3145	0.1534	0.0764	0.1512	0.2907	6.0017	0.1733	0.0863	0.1709	0.3291	7.7045	0.2224	0.1108	0.2196	0.4239
	0	-0.5	5.5760	0.1610	0.0803	0.1592	0.3084	6.2423	0.1802	0.0899	0.1783	0.3461	7.8063	0.2253	0.1124	0.2232	0.4343
		0	5.9841	0.1727	0.0860	0.1698	0.3243	6.7388	0.1945	0.0969	0.1914	0.3662	8.5971	0.2482	0.1236	0.2445	0.4691
		0.5	6.6397	0.1917	0.0952	0.1868	0.3493	7.5474	0.2179	0.1083	0.2126	0.3983	9.8686	0.2849	0.1416	0.2783	0.5232
	0.5	-0.5	6.6624	0.1923	0.0957	0.1885	0.3573	7.4873	0.2161	0.1075	0.2121	0.4029	9.5095	0.2745	0.1366	0.2697	0.5141
		0	7.2577	0.2095	0.1040	0.2036	0.3779	8.2260	0.2375	0.1179	0.2310	0.4299	10.6863	0.3085	0.1532	0.3006	0.5614
		0.5	8.0002	0.2309	0.1143	0.2218	0.4015	9.1506	0.2642	0.1307	0.2541	0.4610	12.1479	0.3507	0.1737	0.3379	0.6154
II	-0.5	0	18.9596	0.5473	0.2712	0.5285	0.9675	21.7315	0.6273	0.3109	0.6060	1.1098	28.3819	0.8193	0.4060	0.7912	1.4478
		0.5	23.2871	0.6722	0.3311	0.6351	1.1120	26.5254	0.7657	0.3773	0.7242	1.2705	34.2015	0.9873	0.4866	0.9347	1.6418
	0	-0.5	21.7646	0.6283	0.3106	0.6015	1.0813	24.9838	0.7212	0.3565	0.6906	1.2421	32.7162	0.9444	0.4668	0.9038	1.6230
		0	26.0797	0.7529	0.3697	0.7040	1.2110	29.7511	0.8588	0.4219	0.8041	1.3855	38.4637	1.1104	0.5456	1.0404	1.7939
		0.5	30.0688	0.8680	0.4232	0.7918	1.3064	34.1688	0.9864	0.4812	0.9017	1.4918	43.8251	1.2651	0.6177	1.1593	1.9227
	0.5	-0.5	28.8839	0.8338	0.4082	0.7713	1.3032	32.9955	0.9525	0.4665	0.8821	1.4928	42.7632	1.2345	0.6047	1.1439	1.9359
		0	32.9048	0.9499	0.4614	0.8561	1.3889	37.4425	1.0809	0.5255	0.9763	1.5879	48.1396	1.3897	0.6761	1.2581	2.0497
		0.5	36.7318	1.0604	0.5108	0.9304	1.4524	41.6763	1.2031	0.5803	1.0591	1.6586	53.2666	1.5377	0.7426	1.3591	2.1353
III	-0.5	0	42.7249	1.2334	0.6042	1.1435	1.9375	49.0844	1.4169	0.6942	1.3138	2.2264	64.3256	1.8569	0.9094	1.7193	2.9061
		0.5	53.7866	1.5527	0.7508	1.3781	2.1855	61.3479	1.7710	0.8569	1.5749	2.5038	79.2245	2.2870	1.1071	2.0368	3.2437
	0	-0.5	48.4501	1.3986	0.6818	1.2747	2.1031	55.7061	1.6081	0.7839	1.4657	2.4182	73.0947	2.1101	1.0280	1.9195	3.1576
		0	59.5707	1.7197	0.8263	1.4965	2.3157	68.0039	1.9631	0.9439	1.7119	2.6555	87.9488	2.5389	1.2213	2.2169	3.4437
		0.5	69.2079	1.9979	0.9459	1.6631	2.4314	78.6545	2.2706	1.0766	1.8978	2.7887	100.8113	2.9102	1.3822	2.4438	3.6123
	0.5	-0.5	65.3466	1.8864	0.9004	1.6086	2.4321	74.6541	2.1551	1.0294	1.8416	2.7911	96.6718	2.7907	1.3336	2.3876	3.6228
		0	75.0806	2.1674	1.0183	1.7654	2.5244	85.3944	2.4651	1.1600	2.0164	2.8987	109.5943	3.1637	1.4914	2.6000	3.7602
		0.5	83.9766	2.4242	1.1209	1.8891	2.5442	95.2018	2.7482	1.2738	2.1551	2.9338	121.3824	3.5040	1.6293	2.7702	3.8237

* for general value of h_0

Table 5.10
Comparison of frequency parameter Ω for homogeneous ($\mu = 0.0$, $\eta = 0.0$) uniform thickness($\alpha = 0.0$, $\beta = 0.0$) Mindlin circular plates

h_0/a	Clamped			S-S			Free		
	I	II	III	I	II	III	I	II	III
0.001	10.2158 10.216* 10.216°	39.7708 39.771* 39.771°	89.1024 89.102* 89.104°	4.9351 4.9351* 4.935°	29.7198 29.720* 29.720°	74.1550 74.155* 74.156°	9.0031 9.0031* 9.003°	38.4429 38.443* 38.443°	87.7489 87.749* 87.750°
0.05	10.1447 10.145*	38.8554 38.855*	84.9950 84.995*	4.9247 4.9247*	29.3233 29.323*	71.7563 71.756*	8.9686 8.9686*	37.7874 37.787*	84.4430 84.443*
0.1	9.9408 9.9408*	36.4787 36.479*	75.6643 75.664*	4.8938 4.8938*	28.2400 28.240*	65.9424 65.942*	8.8679 8.8679*	36.0407 36.401*	76.6756 76.676*
0.15	9.6286 9.6286*	33.3934 33.393*	65.5507 65.551*	4.8440 4.8440*	26.7148 26.715*	59.0621 59.062*	8.7095 8.7095*	33.6744 33.674*	67.8274 67.827*
0.2	9.2400 9.2400*	30.2107 30.211*	56.6823 56.682*	4.7773 4.7773*	24.9945 24.994*	52.5139 52.514*	8.5051 8.5051*	31.1106 31.111*	59.6450 59.645*
0.25	8.8068 8.8068* 8.807°	27.2529 27.253* 27.253°	49.4204 49.420* 49.420°	4.6963 4.6963* 4.696°	23.2541 23.254* 23.254°	46.7745 46.775* 46.775°	8.2674 8.2674* 8.267°	28.6055 28.605* 28.605°	52.5842 52.584* 52.584°

* values taken from Liew et al.[1997].
° Exact values taken from Irie et al.[1980].

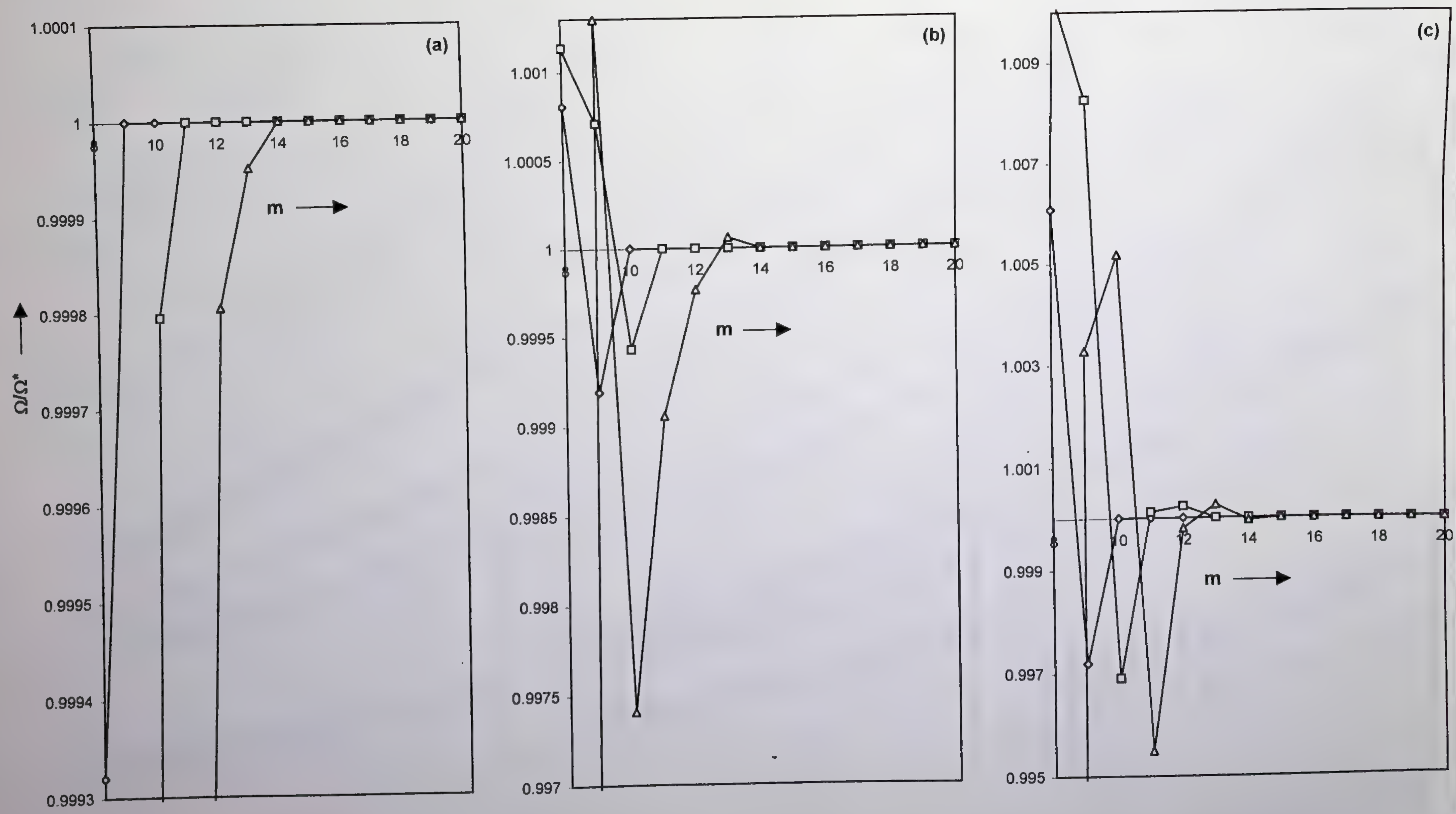

Fig. 5.1 : Convergence of the normalized frequency parameter Ω/Ω^* for the first three modes of vibration with grid refinement for $h_0 = 0.05$, $\eta = -0.5$, $\mu = 1.0$, $\alpha = 0.0$, $\beta = 0.5$ for (a) Clamped (b) Simply Supported and (c) Free plate. ——◊——, Fundamental mode: ——□——, Second mode; ——Δ——, Third mode. Ω^*- the results using 20 collocation points.

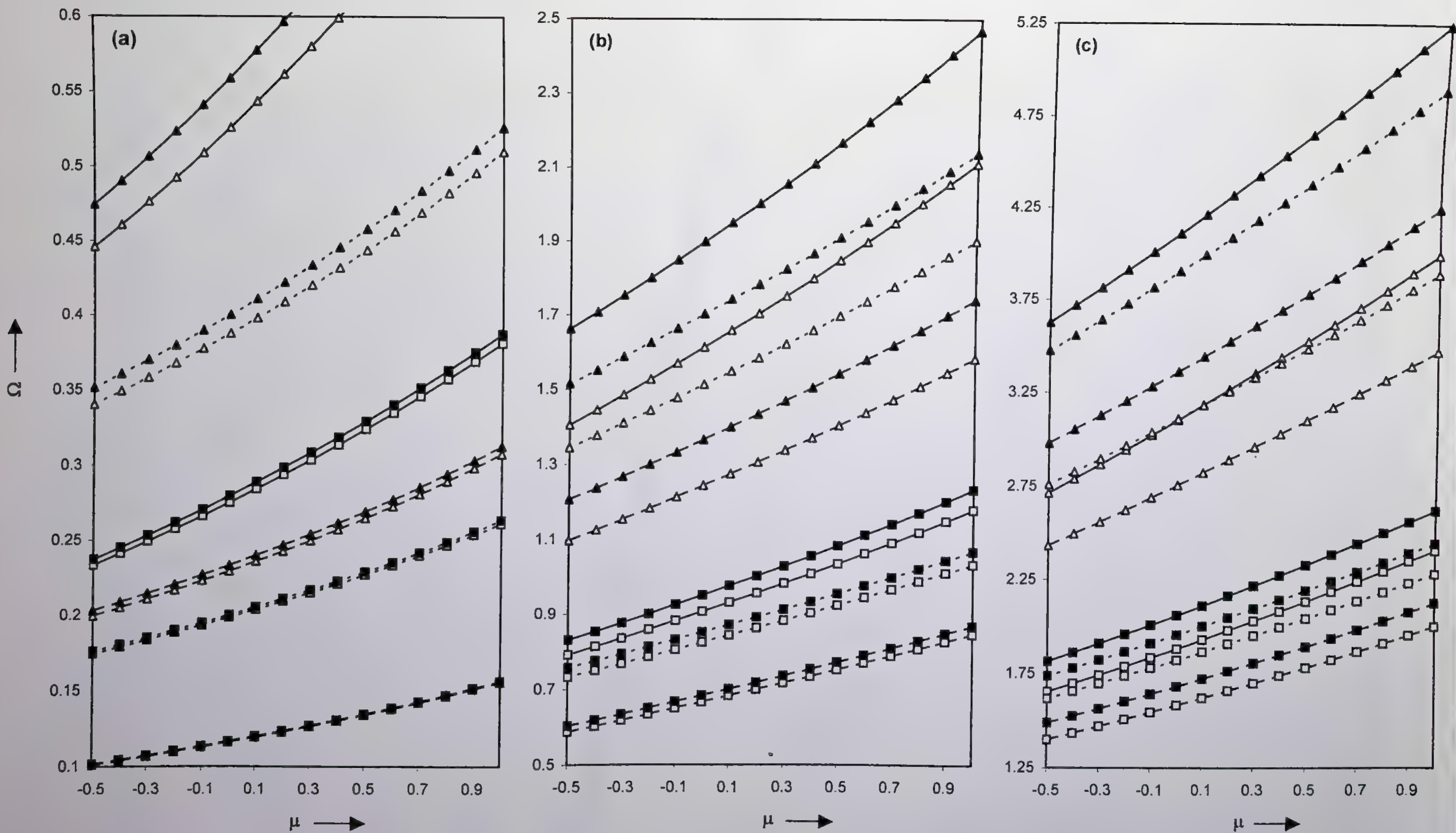

Fig. 5.2 : Frequency parameter for the three plates for $\eta = -0.5$, $\alpha = 0.5$, $\beta = 0.5$ for (a) fundamental (b) second and (c) third mode.
————, clamped ; – – – – – –, simply supported ; -----------, free.
□, $h_0 = 0.05$; Δ, $h_0 = 0.1$. □, Δ, Mindlin plate theory ; ■, ▲, classical plate theory.

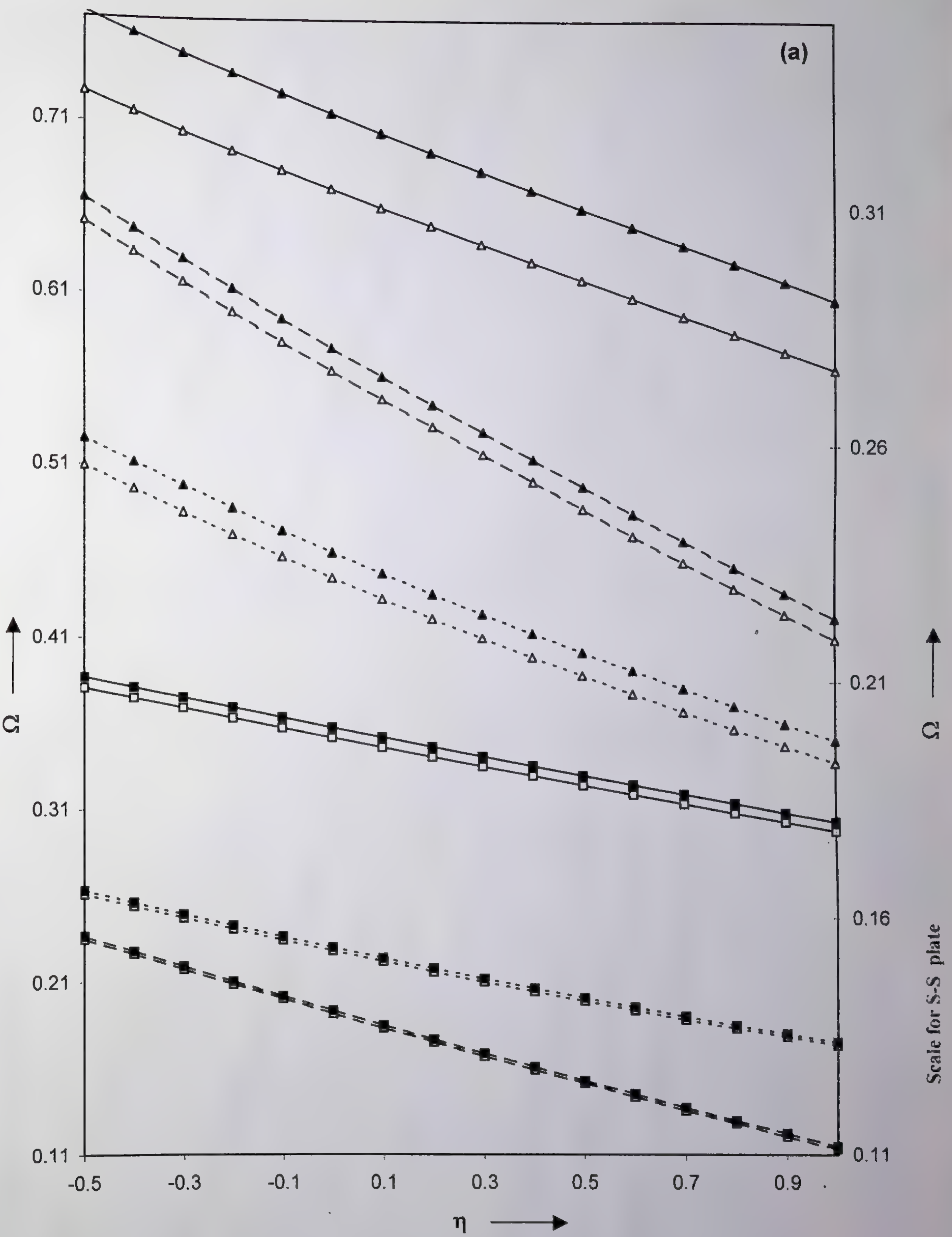

Fig. 5.3a : Frequency parameter for the three plates for $\mu = 1.0$, $\alpha = 0.5$, $\beta = 0.5$ for fundamental mode.
————, clamped ; – – – – – –, simply supported ; -----------, free.
□, $h_0 = 0.05$; Δ, $h_0 = 0.1$. □, Δ, Mindlin plate theory ; ■, ▲, classical plate theory.

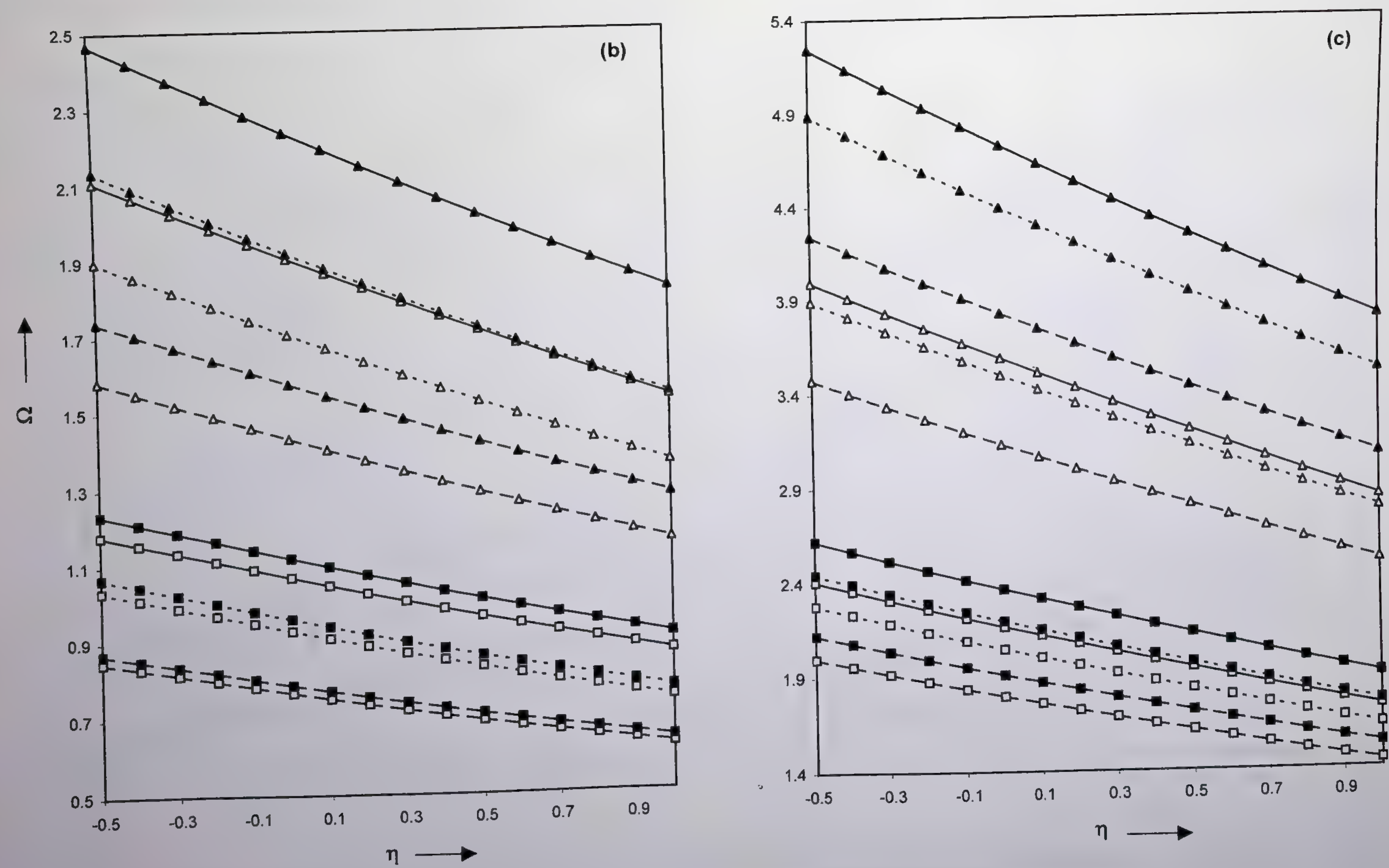

Fig. 5.3 : Frequency parameter for the three plates for $\mu = 1.0$, $\alpha = 0.5$, $\beta = 0.5$ for (b) second and (c) third mode.
———, clamped ; – – – – – –, simply supported ; -----------, free.
□, $h_0 = 0.05$; Δ, $h_0 = 0.1$. □, Δ, Mindlin plate theory ; ■, ▲, classical plate theory.

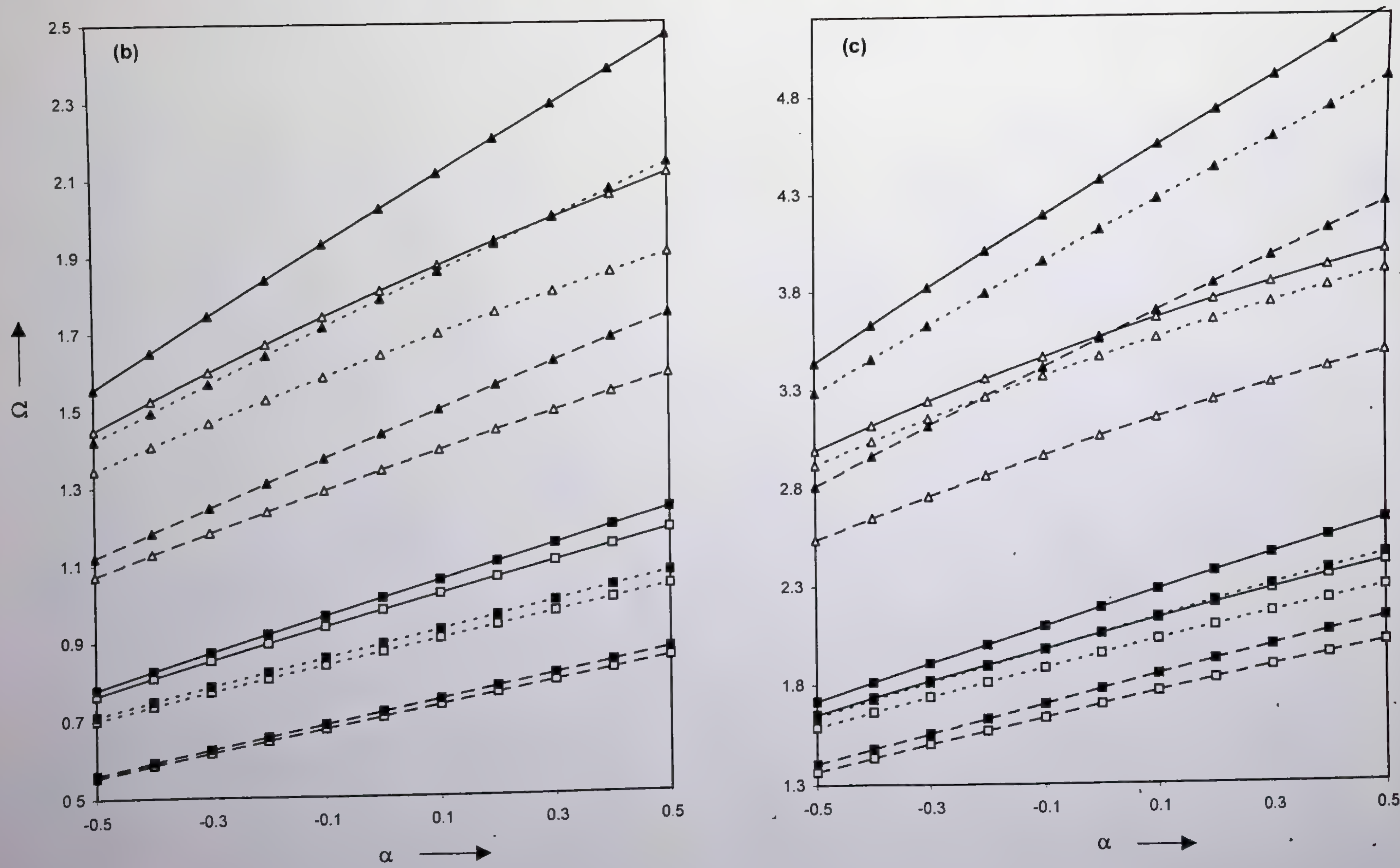

Fig. 5.4 : Frequency parameter for the three plates for $\mu = 1.0$, $\eta = -0.5$, $\beta = 0.5$ for (b) second and (c) third mode.

———, clamped ; – – – – – –, simply supported ; -----------, free.

□, $h_0 = 0.05$; Δ, $h_0 = 0.1$. □, Δ, Mindlin plate theory ; ■, ▲, classical plate theory.

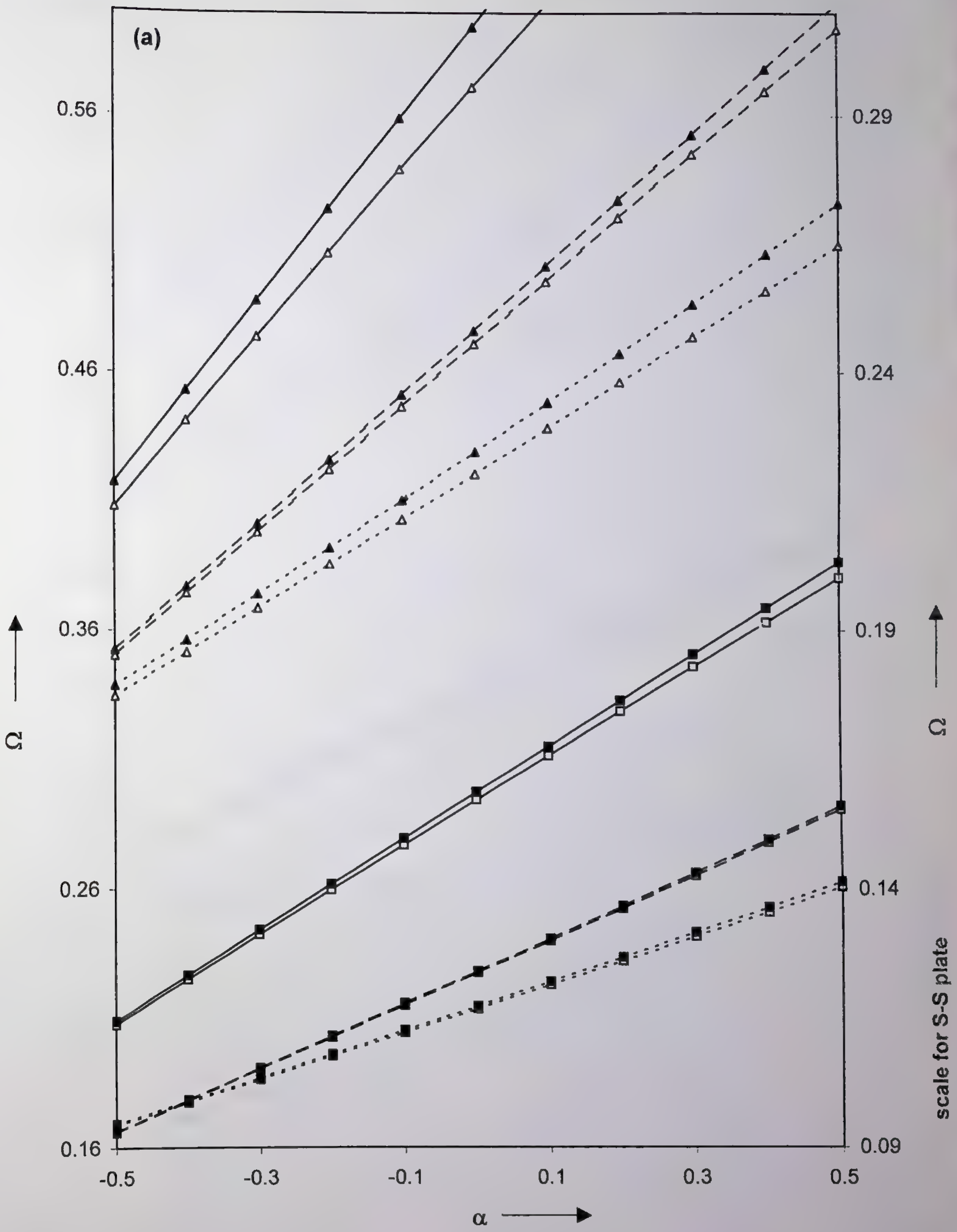

Fig. 5.4a : Frequency parameter for the three plates for $\mu = 1.0$, $\eta = -0.5$, $\beta = 0.5$ for fundamental mode
————, clamped ; – – – – – –, simply supported ; -----------, free.
□, $h_0 = 0.05$; Δ, $h_0 = 0.1$. □, Δ, Mindlin plate theory ; ■, ▲, classical plate theory.

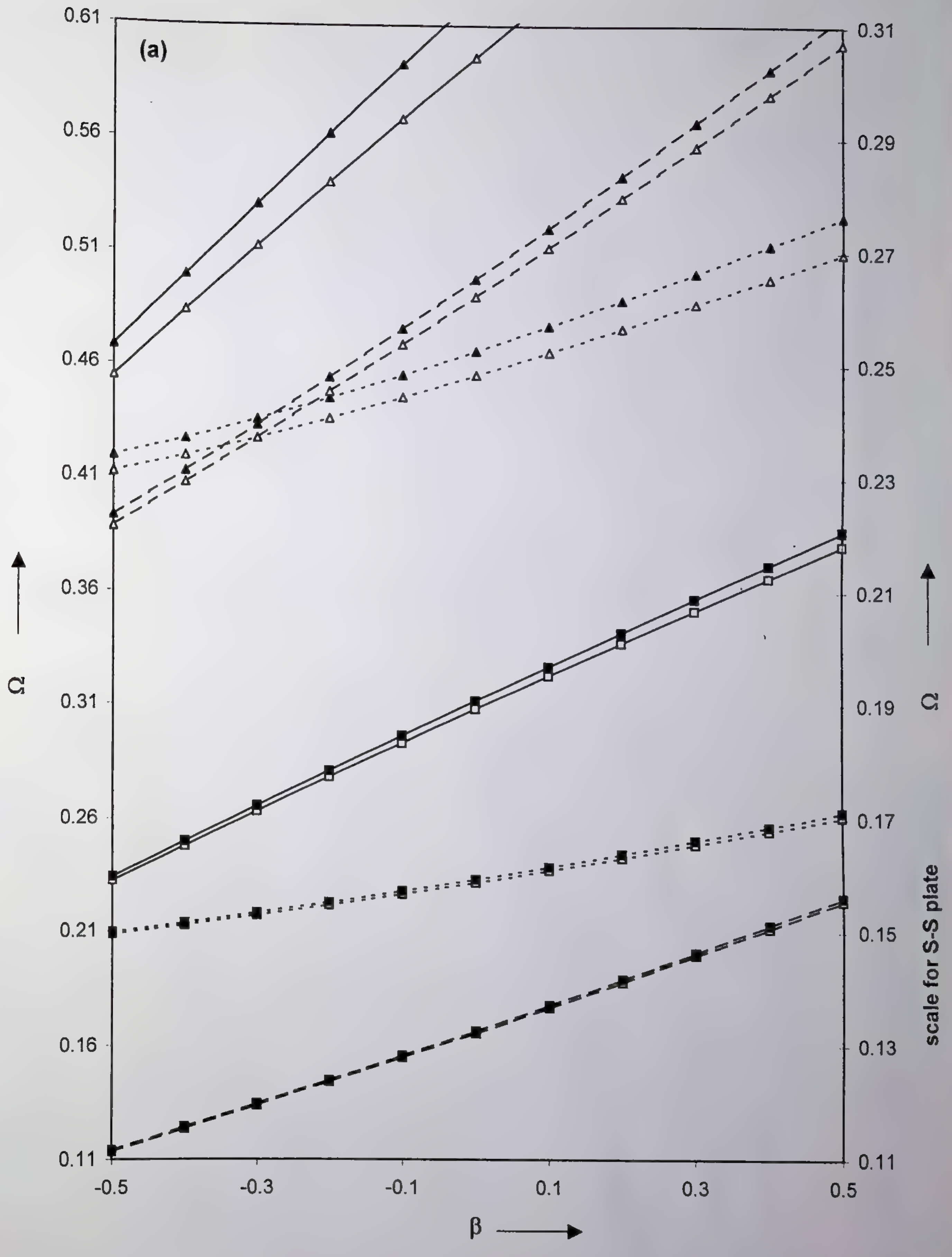

Fig. 5.5a : Frequency parameter for the three plates for $\mu = 1.0$, $\eta = -0.5$, $\alpha = 0.5$ for fundamental mode.
————, clamped ; – – – – – –, simply supported ; -----------, free.
□, $h_0 = 0.05$; Δ, $h_0 = 0.1$. □, Δ, Mindlin plate theory ; ■, ▲, classical plate theory.

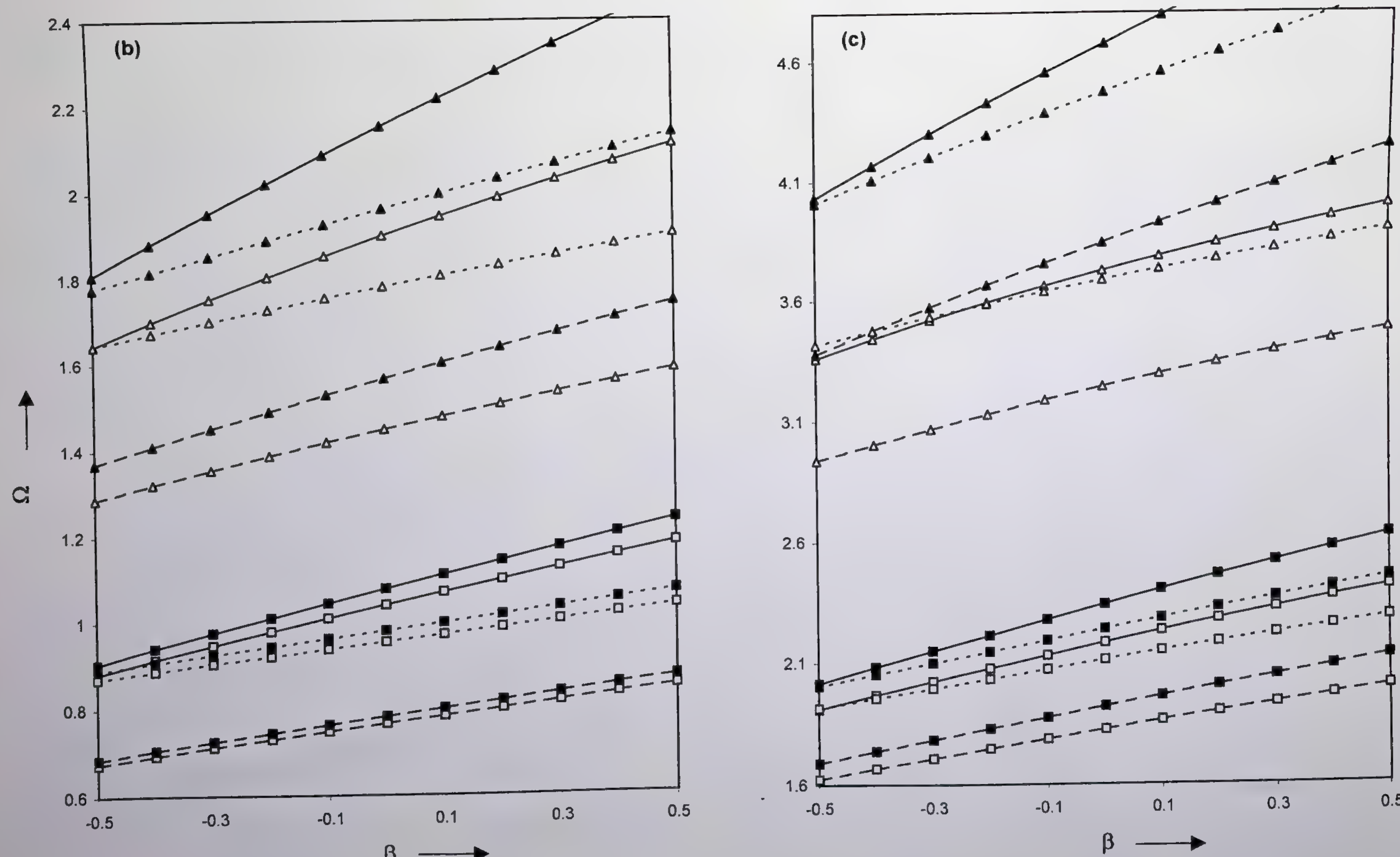

Fig. 5.5 : Frequency parameter for the three plates for $\mu = 1.0$, $\eta = -0.5$, $\alpha = 0.5$ for (b) second and (c) third mode.
————, clamped ; – – – – – –, simply supported ; -----------, free.
□, $h_0 = 0.05$; Δ, $h_0 = 0.1$. □, Δ, Mindlin plate theory ; ■, ▲, classical plate theory.

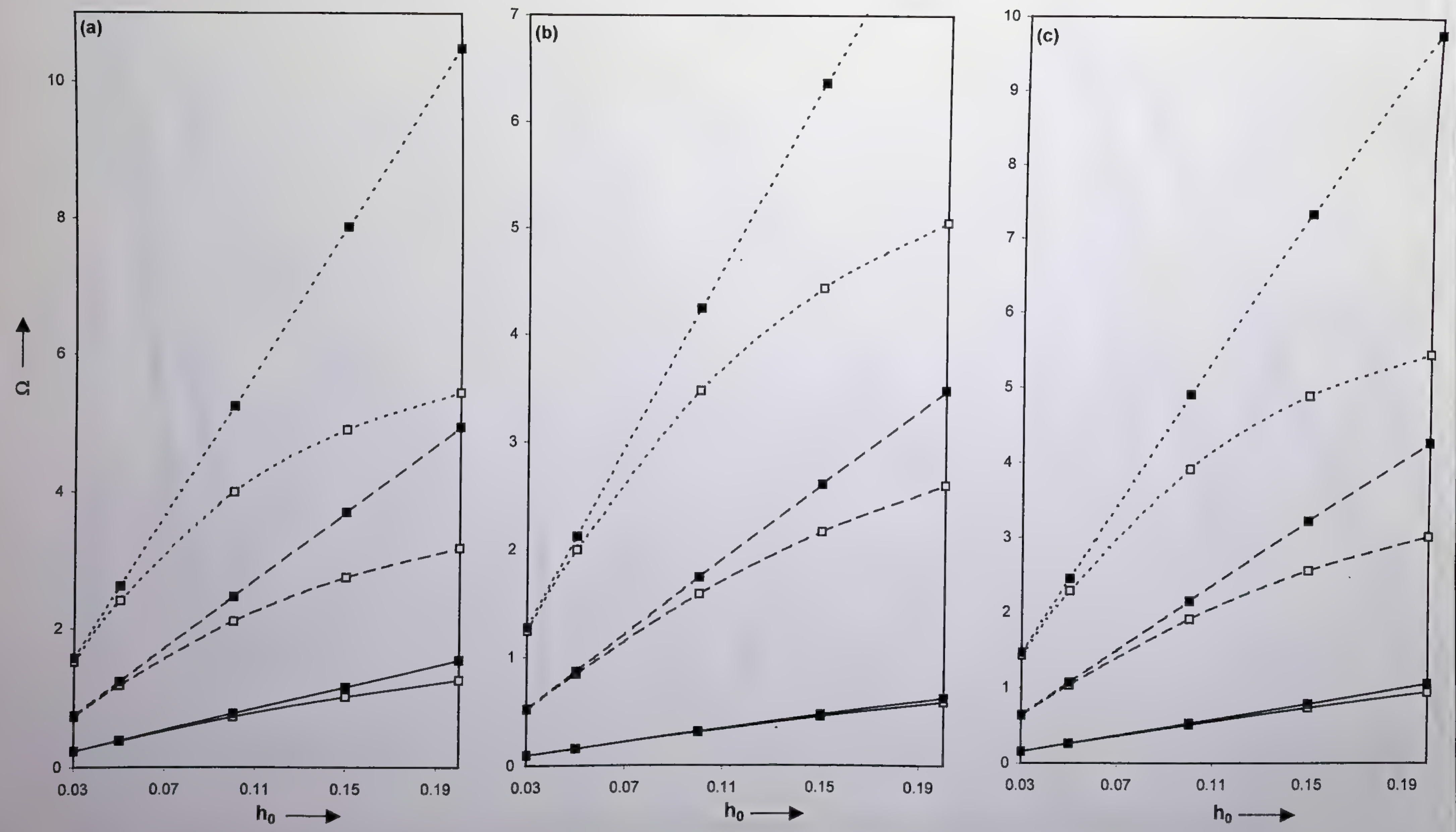

Fig. 5.6 : Frequency parameter for first three modes of vibration for (a) Clamped (b) simply supported and (c) free plate for $\mu = 1.0$, $\eta = -0.5$, $\alpha = 0.5$ and $\beta = 0.5$.

——, fundamental mode; – – – – –, second mode: --------, third mode. □, Mindlin plate theory: ■, classical theory.

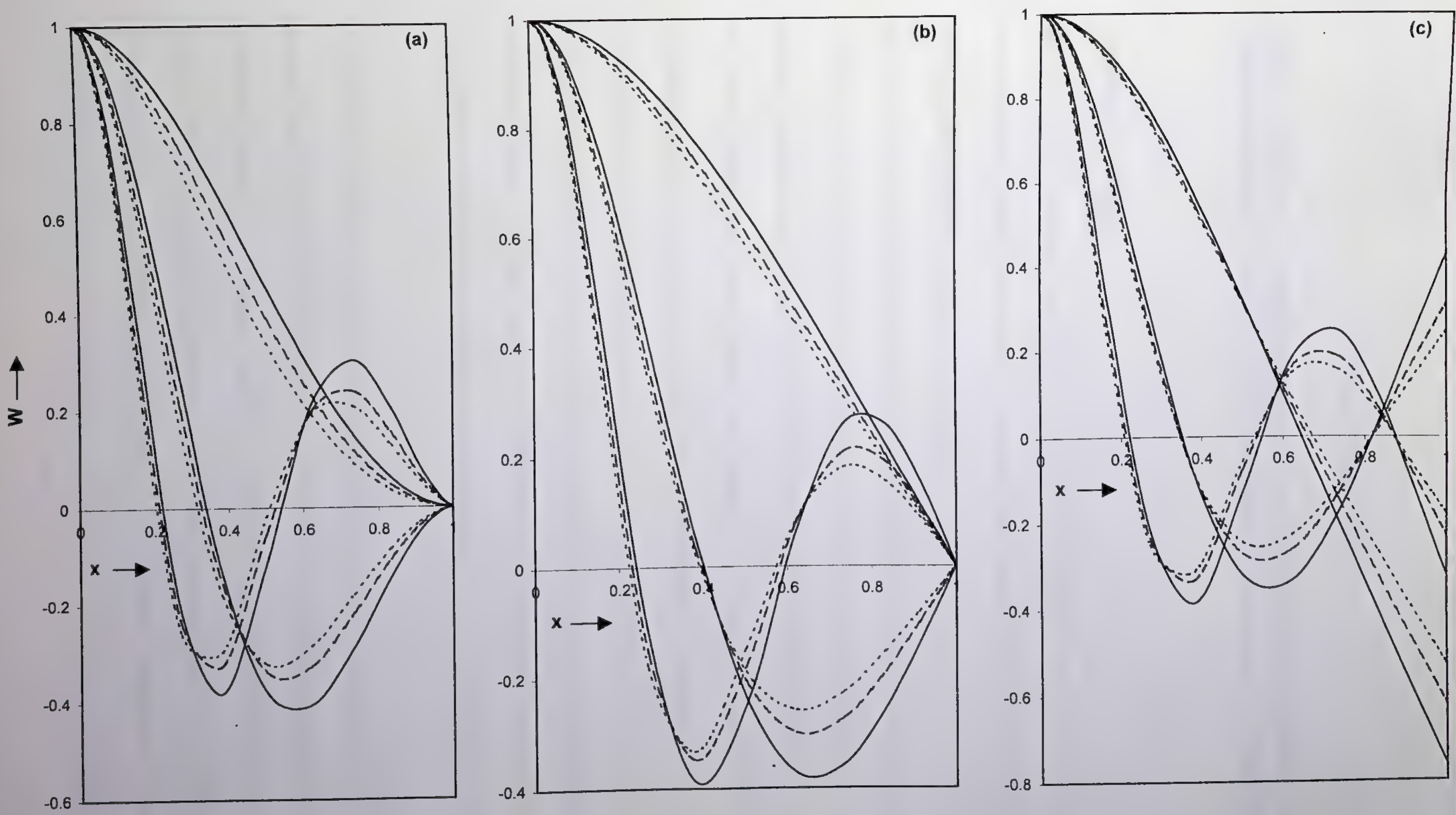

Fig. 5.7 : Normalized displacements for the first three modes of vibration for (a) Clamped (b) Simply Supported and (c) Free plate for $\mu = 1.0$, $\eta = -0.5$. and $h_0 = 0.1$. ———, $\alpha = 0.0$, $\beta = 0.0$; — — — —, $\alpha = 0.5$, $\beta = 0.0$; ---------, $\alpha = 0.5$, $\beta = 0.5$.

CHAPTER VI

AXISYMMETRIC VIBRATIONS OF NON-HOMOGENEOUS POLAR ORTHOTROPIC ANNULAR PLATES OF VARIABLE THICKNESS

1. INTRODUCTION

Plates exhibiting anisotropic characteristics, which were sparingly used a few years ago, now have considerable uses in a wide range of industrial applications. Missile and aircraft designers, solid state physicists and, in general, people engaged in material science, all deal with a variety of anisotropic materials. The development of fibre-reinforced materials and its increasing use in various technological situations (e.g. diaphragms used in pressure capsules, circular plates stiffened with radial and circumferencial ribs, and plates fabricated out of modern composites, viz. boron epoxy, glass epoxy, kevalar and graphites etc.) have necessitated the study of vibrational behaviour of anisotropic plates. The consideration of thickness variation together with orthotropy in structural components, not only ensures reduction in size and weight whilst maintaining high strength, but also meets the desirability of economy. The use of such plates as structural elements in high temperature environmental conditions, demands that the non-homogeneity of the materials should be taken into account for the analysis of plate vibration.

This chapter analyses free axisymmetric vibrations of non-homogeneous polar orthotropic annular plate of quadratically varying thickness. Various numerical techniques, such as finite difference method, finite element method, polynomial coordinate functions, quintic spline method, Chebyshev collocation method etc. have been employed for the solution of governing differential equation of homogeneous plates. The DQ method introduced by Bellman and Casti[1971] for the solution of partial differential equation was promoted by Bert et al.[1988]

and Striz et al.[1988] to solve structural problems. Since then, DQ method has been applied in the area of vibrations by various researchers.

To solve fourth order differential equations by DQ method, Bert and his co-workers[1988] introduced δ-method, in which two boundary conditions were applied at the boundary point as well as at the point apart from the boundary point by a small distance δ. To apply the boundary conditions without the usage of the δ-method, Wang et al.[2003] proposed a New-version Differential Quadrature Method(NDQM) introducing two degrees of freedom for the boundary points for anisotropic rectangular plates and skew plates for a fourth order differential equation. Following Wang et al.[2003, 2004], in this investigation new-version differential quadrature method has been used to determine the first three natural frequencies and mode shapes of annular plates for various values of taper parameters, rigidity ratio, radii ratio and non-homogeneity parameters for three different combinations of boundary conditions. A comparison of results with DQM has also been presented.

2. BASIC PLATE EQUATION

Consider an annular plate of thickness $h(r)$ referred to cylindrical polar coordinates (r, θ, z), where the axis of the plate is taken as the line $r = 0$ and $z = 0$ is the middle surface, shown in Figure 6. Let b and a be the inner and outer radii of the plate, respectively.

Strain Displacement Relations

Let (u, v, w) be the displacement components at a point (r, θ, z) in r, θ and z directions, respectively. We assume that u and v are proportional to z and w is independent of z. For

axisymmetric vibrations, $\frac{\partial}{\partial\theta}=0$. The displacement and strain components are the same as given by relations (2.2.1)-(2.2.2).

Stress-Strain Relations

For an orthotropic material, the stress-strain relations are given by

$$\begin{aligned}
&\sigma_r=\frac{E_r}{(1-\upsilon_r\upsilon_\theta)}\left[\varepsilon_r+\upsilon_\theta\varepsilon_\theta\right],\\
&\sigma_\theta=\frac{E_\theta}{(1-\upsilon_r\upsilon_\theta)}\left[\varepsilon_\theta+\upsilon_r\varepsilon_r\right],\\
&\sigma_{rz}=0,\ \ \sigma_{r\theta}=0,
\end{aligned}\tag{6.2.1}$$

where E_r, E_θ are the Young's moduli in radial and tangential directions, respectively and υ_r, υ_θ are the Poisson's ratios for the plate material with $E_r\upsilon_\theta=\upsilon_r E_\theta$.

If M_r, $M_{r\theta}$, M_θ denote the moment resultants all per unit length, then

$$(M_r,M_{r\theta},M_\theta)=\int_{-h/2}^{h/2}(\sigma_r,\sigma_{r\theta},\sigma_\theta)z\,dz\,. \tag{6.2.2}$$

Integration after substituting σ_r, σ_θ and $\sigma_{r\theta}$ from (6.2.1) into (6.2.2), leads to

$$\begin{aligned}
&M_r=-D_r\left(\frac{\partial^2 w}{\partial r^2}+\frac{\upsilon_\theta}{r}\frac{\partial w}{\partial r}\right),\\
&M_\theta=-D_\theta\left(\frac{1}{r}\frac{\partial w}{\partial r}+\upsilon_r\frac{\partial^2 w}{\partial r^2}\right),\\
&M_{r\theta}=0,
\end{aligned}\tag{6.2.3}$$

where D_r and D_θ are the flexural rigidities defined by

$$D_r=\frac{E_r h^3}{12(1-\upsilon_r\upsilon_\theta)},\qquad D_\theta=\frac{E_\theta h^3}{12(1-\upsilon_r\upsilon_\theta)}.$$

Energy Variations

The strain energy density is given by

$$dW = \frac{1}{2}\left[\sigma_r \varepsilon_r + \sigma_\theta \varepsilon_\theta + \sigma_{r\theta}\varepsilon_{r\theta} + \sigma_{rz}\varepsilon_{rz}\right]dV\,, \tag{6.2.4}$$

where dV denotes elementary volume.

The total strain energy of the plate is obtained by integrating relation (6.2.4) over the total volume of the plate. This gives

$$W = \frac{1}{2}\int_b^a \int_0^{2\pi} \int_{-h/2}^{h/2} \left(\sigma_r \varepsilon_r + \sigma_\theta \varepsilon_\theta\right) r\, dz\, d\theta\, dr\,. \tag{6.2.5}$$

Substituting the values of $\sigma_r, \sigma_\theta, \varepsilon_r, \varepsilon_\theta$ in equation (6.2.5),

$$W = \frac{1}{2}\int_b^a \int_0^{2\pi} \int_{-h/2}^{h/2} \left\{ \begin{array}{l} \dfrac{E_r}{(1-\upsilon_r\upsilon_\theta)}\left[\left(\dfrac{\partial^2 w}{\partial r^2}\right)^2 + \dfrac{\upsilon_\theta}{r}\dfrac{\partial w}{\partial r}\dfrac{\partial^2 w}{\partial r^2}\right] \\ \quad + \dfrac{E_\theta}{(1-\upsilon_r\upsilon_\theta)}\left[\dfrac{1}{r^2}\left(\dfrac{\partial w}{\partial r}\right)^2 + \dfrac{\upsilon_r}{r}\dfrac{\partial w}{\partial r}\dfrac{\partial^2 w}{\partial r^2}\right] \end{array} \right\} z^2 r\, dz\, d\theta\, dr\,. \tag{6.2.6}$$

Integration with respect to z, leads to

$$W = \frac{1}{2}\int_b^a \int_0^{2\pi} \left\{ \begin{array}{l} D_r\left[\left(\dfrac{\partial^2 w}{\partial r^2}\right)^2 + \dfrac{\upsilon_\theta}{r}\dfrac{\partial w}{\partial r}\dfrac{\partial^2 w}{\partial r^2}\right] \\ \quad + D_\theta\left[\dfrac{1}{r^2}\left(\dfrac{\partial w}{\partial r}\right)^2 + \dfrac{\upsilon_r}{r}\dfrac{\partial w}{\partial r}\dfrac{\partial^2 w}{\partial r^2}\right] \end{array} \right\} r\, d\theta\, dr\,. \tag{6.2.7}$$

The expression for kinetic energy is given by

$$dT = \frac{\rho}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] dV \, . \tag{6.2.8}$$

The total kinetic energy, resulting from the vertical displacement of the elements of the plate, is given by

$$T = \frac{1}{2}\int_b^a \int_0^{2\pi} \int_{-h/2}^{h/2} \rho \left(\frac{\partial w}{\partial t}\right)^2 r \, dz \, d\theta \, dr \, . \tag{6.2.9}$$

Integrating with respect to z,

$$T = \frac{1}{2}\int_b^a \int_0^{2\pi} \rho \, h \left(\frac{\partial w}{\partial t}\right)^2 r \, d\theta \, dr \, . \tag{6.2.10}$$

Taking variations of W and T

$$\delta W = \int_b^a \int_0^{2\pi} \left\{ \begin{aligned} & D_r \left[\frac{\partial^2 w}{\partial r^2}\frac{\partial^2 (\delta w)}{\partial r^2} + \frac{1}{2}\frac{\upsilon_\theta}{r}\left(\frac{\partial w}{\partial r}\frac{\partial^2 (\delta w)}{\partial r^2} + \frac{\partial (\delta w)}{\partial r}\frac{\partial^2 w}{\partial r^2} \right) \right] \\ & + D_\theta \left[\frac{1}{r^2}\frac{\partial w}{\partial r}\frac{\partial (\delta w)}{\partial r} + \frac{1}{2}\frac{\upsilon_r}{r}\left(\frac{\partial w}{\partial r}\frac{\partial^2 (\delta w)}{\partial r^2} + \frac{\partial (\delta w)}{\partial r}\frac{\partial^2 w}{\partial r^2} \right) \right] \end{aligned} \right\} r \, d\theta \, dr \, , \tag{6.2.11}$$

$$\delta T = \int_b^a \int_0^{2\pi} \rho h \left(\frac{\partial w}{\partial t}\frac{\partial (\delta w)}{\partial t} \right) r \, d\theta \, dr \, . \tag{6.2.12}$$

Equation of Motion

To obtain the equations of motion, Hamilton's energy principle is used which can be written as,

$$\delta \int_{t_1}^{t_2} L\,dt = 0\,, \tag{6.2.13}$$

where t_1 and t_2 are the initial and final values of time t and the kinetic potential L is given by

$L = T - W.$

Taking the variational operator δ inside the integral and considering $\delta W - \delta T$, equation (6.2.13) becomes

$$\int_b^a \int_0^{2\pi} \int_{t_1}^{t_2} \left[\begin{array}{l} D_r \left\{ \dfrac{\partial^2 w}{\partial r^2} \dfrac{\partial^2 (\delta w)}{\partial r^2} + \dfrac{1}{2} \dfrac{\upsilon_\theta}{r} \left(\dfrac{\partial w}{\partial r} \dfrac{\partial^2 (\delta w)}{\partial r^2} + \dfrac{\partial (\delta w)}{\partial r} \dfrac{\partial^2 w}{\partial r^2} \right) \right\} \\ + D_\theta \left[\dfrac{1}{r^2} \dfrac{\partial w}{\partial r} \dfrac{\partial (\delta w)}{\partial r} + \dfrac{1}{2} \dfrac{\upsilon_r}{r} \left(\dfrac{\partial w}{\partial r} \dfrac{\partial^2 (\delta w)}{\partial r^2} + \dfrac{\partial (\delta w)}{\partial r} \dfrac{\partial^2 w}{\partial r^2} \right) \right] \\ - \rho h \dfrac{\partial w}{\partial t} \dfrac{\partial (\delta w)}{\partial t} \end{array} \right] r\,dt\,d\theta\,dr = 0\,. \tag{6.2.14}$$

Using $D_r \upsilon_\theta = \upsilon_r D_\theta$,

$$\int_b^a \int_0^{2\pi} \int_{t_1}^{t_2} \left[\begin{array}{l} D_r \left\{ \dfrac{\partial^2 w}{\partial r^2} \dfrac{\partial^2 (\delta w)}{\partial r^2} + \dfrac{1}{r} \upsilon_\theta \left(\dfrac{\partial^2 w}{\partial r^2} \dfrac{\partial (\delta w)}{\partial r} + \dfrac{\partial w}{\partial r} \dfrac{\partial^2 (\delta w)}{\partial r^2} \right) + \dfrac{\upsilon_\theta}{\upsilon_r} \dfrac{1}{r^2} \dfrac{\partial w}{\partial r} \dfrac{\partial (\delta w)}{\partial r} \right\} \\ - \rho h \dfrac{\partial w}{\partial t} \dfrac{\partial (\delta w)}{\partial t} \end{array} \right] r\,dt\,d\theta\,dr = 0\,. \tag{6.2.15}$$

Integrating equation (6.2.15) by parts, the integrated part gives boundary conditions while the remaining triple integrals are

$$\int_b^a \int_0^{2\pi} \int_{t_1}^{t_2} \left[\left\{ \begin{aligned} &\frac{\partial^2}{\partial r^2}\left(D_r r \frac{\partial^2 w}{\partial r^2}\right) - \upsilon_\theta \frac{\partial}{\partial r}\left(D_r \frac{\partial^2 w}{\partial r^2}\right) \\ &+ \upsilon_\theta \frac{\partial^2}{\partial r^2}\left(D_r \frac{\partial w}{\partial r}\right) - \frac{\upsilon_\theta}{\upsilon_r}\frac{\partial}{\partial r}\left(\frac{D_r}{r}\frac{\partial w}{\partial r}\right) \end{aligned} \right\} + \rho h r \frac{\partial^2 w}{\partial t^2} \right] \delta w \, dt \, d\theta \, dr = 0 . \tag{6.2.16}$$

Expression (6.2.16) will be satisfied only when the coefficient of δw is zero and hence,

$$\begin{aligned} &\frac{\partial^2}{\partial r^2}\left(D_r r \frac{\partial^2 w}{\partial r^2}\right) - \upsilon_\theta \frac{\partial}{\partial r}\left(D_r \frac{\partial^2 w}{\partial r^2}\right) + \upsilon_\theta \frac{\partial^2}{\partial r^2}\left(D_r \frac{\partial w}{\partial r}\right) \\ &\qquad - \frac{\upsilon_\theta}{\upsilon_r}\frac{\partial}{\partial r}\left(\frac{D_r}{r}\frac{\partial w}{\partial r}\right) + \rho h r \frac{\partial^2 w}{\partial t^2} = 0, \end{aligned} \tag{6.2.17}$$

which is the required plate equation of motion.

For a non-homogeneous plate, simplification of equation (6.2.17) leads to

$$\begin{aligned} &E_r \frac{\partial^4 w}{\partial r^4} + \frac{2}{r}\left[E_r + r\frac{dE_r}{dr}\right]\frac{\partial^3 w}{\partial r^3} \\ &\quad + \frac{1}{r^2}\left[-E_\theta + r(2+\upsilon_\theta)\frac{dE_r}{dr} + r^2 \frac{d^2 E_r}{dr^2}\right]\frac{\partial^2 w}{\partial r^2} \\ &\qquad + \frac{1}{r^3}\left[E_\theta - r\frac{dE_\theta}{dr} + r^2 \upsilon_\theta \frac{d^2 E_r}{dr^2}\right]\frac{\partial w}{\partial r} \\ &\qquad\qquad + \frac{12(1-\upsilon_r \upsilon_\theta)\rho}{h^2}\frac{\partial^2 w}{\partial t^2} = 0. \end{aligned} \tag{6.2.18}$$

Introducing non-dimensional variables $x=\frac{r}{a}, \bar{w}=\frac{w}{a}, \bar{h}=\frac{h}{a}$, together with quadratic thickness variation along radial direction, i.e. $\bar{h}=h_0\left(1+\alpha x+\beta x^2\right)$ and exponential variation for non-homogeneity of the material in radial direction as follows :

$$E_r = E_1 e^{\mu x}, \qquad E_\theta = E_2 e^{\mu x}, \qquad \rho = \rho_0 e^{\eta x}, \tag{6.2.19}$$

equation (6.2.18) reduces to

$$P_0\frac{d^4W}{dx^4}+P_1\frac{d^3W}{dx^3}+P_2\frac{d^2W}{dx^2}+P_3\frac{dW}{dx}+P_4W=0, \tag{6.2.20}$$

where $\bar{w}(x,t)=W(x)e^{i\omega t}$ (for harmonic vibrations), ω is the radian frequency, h_0, ρ_0 are the thickness and density at the centre of the plate, μ the non-homogeneity parameter, η the density parameter, α, β the taper parameters,

$$P_0=1, \qquad P_1=\frac{2(1+Bx)}{x}, \qquad P_2=B^2+C+\frac{(2+\upsilon_\theta)}{x}B-\frac{p}{x^2},$$

$$P_3=\frac{p}{x^3}(1-Bx)+\frac{\upsilon_\theta}{x}\left(B^2+C\right), \qquad P_4=-\frac{\Omega^2 e^{(\eta-\mu)}}{A^2},$$

$$p=\frac{E_2}{E_1}, \qquad \Omega^2=\frac{12\rho_0 a^2\omega^2\left(1-\upsilon_r\upsilon_\theta\right)}{E_1 h_0^2},$$

$$A=1+\alpha x+\beta x^2, \qquad B=\mu+\frac{3(\alpha+2\beta x)}{A}, \qquad C=\frac{3\left(2\beta-\alpha^2-2\beta^2x^2-2\alpha\beta x\right)}{A^2}$$

and E_1, E_2 are Young's moduli in radial and tangential directions at $x = 0$ respectively.

Equation (6.2.20) together with boundary conditions at the edges $x = \varepsilon$ and $x = 1$, where $\varepsilon = b/a$, constitutes a two-point boundary value problem in the range $(\varepsilon, 1)$ which has been solved by new version of DQM.

3. METHOD OF SOLUTION : NDQM

Let $x_1, x_2, \ldots, x_m$ be the m grid points in the applicability range $[\varepsilon, 1]$ of the plate. According to new version differential quadrature method (Wang et al.[2003]), first, second, third and fourth order derivatives of $W(x)$ with respect to x can be expressed discretely at the point x_i as

$$\frac{dW(x_i)}{dx} = \sum_{j=1}^{m+2} \bar{A}_{ij}\,\delta_j, \qquad \frac{d^2W(x_i)}{dx^2} = \sum_{j=1}^{m+2} \bar{B}_{ij}\,\delta_j,$$
$$\frac{d^3W(x_i)}{dx^3} = \sum_{j=1}^{m+2} \bar{C}_{ij}\,\delta_j, \qquad \frac{d^4W(x_i)}{dx^4} = \sum_{j=1}^{m+2} \bar{D}_{ij}\,\delta_j, \qquad i = 2, 3, \ldots, (m\text{-}1)$$

where $\delta_j = W_j \equiv W(x_j),\ j = 1,2,\ldots,m;\ \delta_{m+1} = W_1' \equiv W'(x_1);\ \delta_{m+2} = W_m' \equiv W'(x_m)$ and $\bar{A}_{ij}, \bar{B}_{ij}, \bar{C}_{ij}, \bar{D}_{ij}$ are weighting coefficients associated with first four derivatives of $W(x)$, respectively. at discrete point x_i given by

$$\bar{A}_{ij} = \frac{M^{(1)}(x_i)}{(x_i - x_j)M^{(1)}(x_j)} \qquad \text{for } i \neq j,\ i = 1(1)m,\ j = 1(1)m,$$

$$\text{where } M^{(1)}(x_i) = \prod_{\substack{k=1\\k\neq i}}^{m}(x_i - x_k) \quad \text{and} \quad \bar{A}_{ii} = \sum_{\substack{k=1\\k\neq i}}^{m}\frac{1}{(x_i - x_k)}, \qquad i = 1(1)m$$

$$\bar{A}_{i,m+1} = \bar{A}_{i,m+2} = 0\,, \qquad i = 1(1)m$$

$$\bar{B}_{ij} = \sum_{k=1}^{m}\bar{A}_{ik}\bar{A}_{kj}\,, \qquad i = 2(1)m\text{-}1,\ j = 1(1)m$$

$$\bar{B}_{i,m+1} = \bar{B}_{i,m+2} = 0\,, \qquad i = 2(1)m\text{-}1$$

$$\bar{B}_{i,m+1} = \bar{A}_{i1}\,, \quad \bar{B}_{i,m+2} = \bar{A}_{im}\,, \qquad i = 1, m$$

$$\bar{C}_{ij} = \sum_{k=2}^{m-1}\left(\bar{A}_{ik}\bar{B}_{kj} + \bar{A}_{i1}\bar{A}_{1k}\bar{A}_{kj} + \bar{A}_{im}\bar{A}_{mk}\bar{B}_{kj}\right), \qquad i = 1(1)m,\ j = 1(1)m$$

$$\bar{C}_{i,m+1} = \bar{A}_{i1}\bar{A}_{11} + \bar{A}_{im}\bar{A}_{m1}\,, \quad \bar{C}_{i,m+2} = \bar{A}_{i1}\bar{A}_{1m} + \bar{A}_{im}\bar{A}_{mm}\,, \qquad i = 1(1)m$$

$$\bar{D}_{ij} = \sum_{k=2}^{m-1}\left(\bar{B}_{ik}\bar{B}_{kj} + \bar{B}_{i1}\bar{A}_{1k}\bar{A}_{kj} + \bar{B}_{im}\bar{A}_{mk}\bar{B}_{kj}\right), \qquad i = 2(1)m\text{-}1,\ j = 1(1)m$$

$$\bar{D}_{i,m+1} = \bar{B}_{i1}\bar{A}_{11} + \bar{B}_{im}\bar{A}_{m1}\,, \quad \bar{D}_{i,m+2} = \bar{B}_{i1}\bar{A}_{1m} + \bar{B}_{im}\bar{A}_{mm}\,, \qquad i = 2(1)m\text{-}1.$$

Discretizing equation (6.2.20) at node x_i, it reduces to

$$\sum_{j=1}^{m+2}(P_0\overline{D}_{ij}+P_{1,i}\overline{C}_{ij}+P_{2,i}\overline{B}_{ij}+P_{3,i}\overline{A}_{ij})\delta_j+P_{4,i}W(x_i)=0 \ , \qquad \text{for } i=2, 3,\ldots, (m\text{-}1) \qquad (6.3.1)$$

Satisfaction of equation (6.3.1) at (m-2) nodal points x_i , i = 2, 3, ….., (m-1) provides a set of (m-2) equations in terms of unknowns $\delta_j, j=1, 2,\ldots,(m+2)$. The system of equations can be written in the matrix form as

$$[B][\delta^*]=[0] \quad , \qquad (6.3.2)$$

where B and δ^* are matrices of order (m-2) x (m+2) and (m+2) x 1, respectively.

The (m-2) grid points, chosen for collocation, are taken as the zeros of shifted Chebyshev polynomial of order (m-2) with orthogonality range (ε,1). The choice of grid points is based upon the fact that zeros of shifted Chebyshev polynomial provide a faster rate of convergence as was seen in chapter II.

4. BOUNDARY CONDITIONS AND FREQUENCY EQUATIONS

The three sets of boundary conditions i.e. C-C, C-S and C-F have been considered here. By satisfying the relations:

(i) $W=\dfrac{dW}{dx}=0$

(ii) $W=\dfrac{d^2W}{dx^2}+\dfrac{\upsilon_\theta}{x}\dfrac{dW}{dx}=0$ (6.4.1)

(iii) $\dfrac{d^2W}{dx^2}+\dfrac{\upsilon_\theta}{x}\dfrac{dW}{dx}=\dfrac{d^3W}{dx^3}+\dfrac{1}{x}\dfrac{d^2W}{dx^2}-\dfrac{p}{x^2}\dfrac{dW}{dx}=0$

for clamped, simply supported and free edge conditions, respectively, a set of four homogeneous equations in terms of δ_j are obtained. These equations together with field equations (6.3.2) give a complete set of (m+2) equations in (m+2) unknowns. For a C-C plate, the above set of homogeneous equations can be written as

$$\begin{bmatrix} B \\ B^{CC} \end{bmatrix} \left[\delta^{*} \right] = [0] , \tag{6.4.2}$$

where B^{CC} is a matrix of order 4 x (m+2).

For a non-trivial solution of equation (6.4.2), the frequency determinant must vanish and hence

$$\begin{vmatrix} B \\ B^{CC} \end{vmatrix} = 0 \ . \tag{6.4.3}$$

Similarly for C-S and C-F plates, the frequency determinants can respectively be written as

$$\begin{vmatrix} B \\ B^{CS} \end{vmatrix} = 0 , \qquad \begin{vmatrix} B \\ B^{CF} \end{vmatrix} = 0 \ . \tag{6.4.4, 6.4.5}$$

5. NUMERICAL RESULTS & DISCUSSION

The frequency equations (6.4.3-6.4.5) have been solved to obtain the values of the frequency parameter Ω. In the present work, the first three natural frequencies of vibration have been computed for three different combinations of boundary conditions for non-homogeneity parameter μ = -0.5(0.1)1.0; density parameter η = -0.5(0.1)1.0; radii ratio ε = 0.3(0.05)0.7; rigidity ratio p = 0.5(0.25)5.0 and taper parameters α = -0.5(0.1)0.5; β = -0.5(0.1)0.5 such that $\alpha+\beta > -1$ for $\upsilon_\theta = 0.3$.

Figures 6.1(a,b,c) show the convergence of the frequency parameter Ω with the increasing number of grid points for a specified plate for three sets of boundary conditions, respectively. During computation, the value of m has been fixed as 19, since there was no further improvement even at fourth place of decimal.

Numerical results for specified plate parameters are given in Tables (6.1-6.18) and Figures (6.2-6.8). Tables (6.1-6.18) present the values of frequency parameter Ω for different values of plate parameters, namely non-homogeneity parameter μ (= -0.5, 0.0, 1.0), density parameter η (= -0.5, 0.0, 1.0), rigidity ratio parameter p (= 0.5, 1.0, 2.0, 5.0), taper parameters α (= -0.5. 0.0, 0.5); β (= -0.5, 0.0, 0.5) such that $\alpha + \beta > -1$ for radii ratio ε (= 0.3, 0.5) for C-C, C-S and C-F plates, respectively. From the results, it is found that the frequency parameter increases with increasing value of radii ratio ε and parameters, namely non-homogeneity μ, rigidity ratio p, taper α and β while, it decreases with increasing value of η. Also, the frequency parameter for C-S plate is smaller than that for C-C plate and greater than that for C-F plate irrespective of the values of other plate parameters.

Figures 6.2(a,b,c) show the plots of frequency parameter Ω versus non-homogeneity parameter μ for fixed values of density parameter $\eta = 1.0$, rigidity parameter $p = 5.0$, radii ratio $\varepsilon = 0.3$ and taper parameters $\alpha = -0.3, 0, 0.3$; $\beta = -0.3, 0, 0.3$ for three sets of boundary conditions vibrating in fundamental, second and third mode, respectively. It is observed that frequency parameter increases with increasing values of non-homogeneity parameter μ for all the three plates. However, the rate of increase for C-S plate is lower as compared to that for C-C plate and higher as compared to that for C-F plate, keeping all the plate parameters fixed. The rate of

increase with increasing value of μ increases with increase in number of modes. The frequency parameter increases with increase in taper parameters α as well as β.

Figures 6.3(a,b,c) show the plots of first three frequency parameters Ω with varying values of density parameter η for non-homogeneity parameter μ = 1.0, rigidity parameter p = 5.0, radii ratio ε = 0.3, and taper parameters α = -0.3, 0, 0.3; β = -0.3, 0, 0.3 for C-C, C-S and C-F plates respectively. It is observed that frequency parameter decreases with increasing values of density parameter η for all the three plates. However, the rate of decrease is pronounced in order of plates C-F, C-S, C-C, keeping all other plate parameters fixed. This rate of decrease in third mode is higher than that in second mode and that in second mode is higher than that in the fundamental mode. Figures 6.4(a,b,c) depict the effect of rigidity parameter p on frequency parameter Ω for fixed values of non-homogeneity parameter μ = 1.0, density parameter η = -0.5, radii ratio ε = 0.3 and taper parameters α = -0.3, 0, 0.3; β = -0.3, 0, 0.3 for all the three plates vibrating in fundamental, second and third mode respectively. The frequency parameter is found to increase as the plate becomes more and more stiff in tangential direction ($p > 1$) as compared to radial direction ($p < 1$). The rate of change of Ω with increasing values of p in case of C-C plate, is greater as compared to that of C-S plate and that of C-S plate is greater than that of C-F plate for a fixed set of values of all other plate parameters.

Figure 6.5 shows the behaviour of frequency parameter Ω with radii ratio ε for circumferentially stiffened plate for μ = 1.0, η = -0.5, α = -0.3, 0.3; β = -0.3, 0.3 vibrating in fundamental mode. The frequency parameter increases with increase in hole size of plate. The rate of increase is pronounced for $\varepsilon > 0.5$. Also, the frequency can be increased/decreased by

increasing/decreasing taper parameters α and β. However, the effect of parameter β is lower than that of α.

Figures 6.6(a,b,c) depict the variation of frequency parameter Ω with taper parameter α for fixed values of parameters $\mu = 1.0$, $\eta = -0.5$, $\varepsilon = 0.3$, $p = 1.0, 5.0$ and $\beta = -0.3, 0.3$ for all the three plates for fundamental, second and third mode respectively. It is observed that frequency parameter Ω increases with increasing value of α except for isotropic C-F plate for $\beta = -0.3$. In this case, frequency first decreases and then increases with a local minima in the vicinity of $\alpha = -0.1$. The rate of increase of Ω with increase in α is pronounced in C-C plate as compared to that in C-S and C-F plates. Also, this rate of increase for circumferentially stiffened plate is higher than that for isotropic plate. The rate of increase increases with number of modes. Figures 6.7(a,b,c) show the plots of frequency parameter Ω versus taper parameter β for C-C, C-S and C-F plates for fixed values of parameters $\mu = 1.0$, $\eta = -0.5$, $\varepsilon = 0.3$, $p = 1.0, 5.0$ and $\alpha = -0.3, 0.3$ vibrating in fundamental, second and third mode respectively. The frequency parameter Ω is found to increase with increasing value of α except for C-F plate. In this case, for $\beta = -0.3$ the frequency first decreases and then increases with a local minima in the vicinity of $\alpha = 0.1$ and this minima shifts towards lower values of β as the plate becomes more and more circumferentially stiffened as well as the plate becomes thicker and thicker towards the outer edge. The rate of increase of Ω with increase in β increases with number of modes and in the order C-F, C-S and C-C plate. Figures 6.8(a,b,c) show the normalized displacements for $\mu = 1.0$, $\eta = -0.5$, $\varepsilon = 0.3$, $p = 5.0$, $\alpha = 0, \beta = 0$; $\alpha = 0.3, \beta = 0$ and $\alpha = 0.3, \beta = 0.3$. The nodal circles shift towards the centre as the plate becomes thicker and thicker towards outer edge.

A comparison of minimum number of collocation points used to obtain frequencies with four digit exactitude for specified plates employing DQM and new version of DQM has been presented in Table 6.19, which shows that NDQM converges faster as compared to DQM for C-S and C-F plates, while in case of C-C plate, DQM is little bit faster than NDQM. Table 6.20 shows a comparison of results for homogeneous (μ = 0.0, η = 0.0) isotropic (p = 1) and orthotropic (p = 5) plates of uniform thickness (α = 0.0, β = 0.0) for ε = 0.3 with those of Verma [1987] obtained by quintic spline technique, Gorman [1982] by finite element method. Avalos and Laura [1979] by Galerkin's method, Larrondo et al. [1994] by Rayleigh-Ritz method and exact solutions given by Leissa[1969] for isotropic plates. A close agreement between the results is found, which shows the versatility of the new version of differential quadrature method.

Table 6.1

Values of frequency parameter Ω for C-C plate vibrating in fundamental mode for $\varepsilon = 0.3$

			η								
			-0.5			0.0			1.0		
			μ			μ			μ		
α	β	p	-0.5	0.0	1.0	-0.5	0.0	1.0	-0.5	0.0	1.0
-0.5	0	0.5	29.5740	34.7862	48.0989	24.9914	29.4462	40.8550	17.7546	20.9906	29.3224
		1	29.8639	35.1222	48.5495	25.2335	29.7276	41.2340	17.9224	21.1864	29.5887
		2	30.4327	35.7819	49.4352	25.7084	30.2797	41.9788	18.2512	21.5705	30.1116
		5	32.0586	37.6707	51.9790	27.0651	31.8596	44.1166	19.1892	22.6679	31.6104
	0.5	0.5	41.6363	49.0802	68.1680	35.4073	41.8103	58.2738	25.4734	30.1846	42.3654
		1	41.9983	49.5019	68.7403	35.7121	42.1662	58.7592	25.6880	30.4364	42.7120
		2	42.7109	50.3323	69.8683	36.3120	42.8669	59.7156	26.1101	30.9317	43.3947
		5	44.7636	52.7273	73.1288	38.0390	44.8869	62.4792	27.3238	32.3581	45.3654
0	-0.5	0.5	32.4792	38.1969	52.7886	27.4184	32.2998	44.7902	19.4384	22.9760	32.0760
		1	32.8105	38.5799	53.2999	27.6949	32.6202	45.2198	19.6296	23.1986	32.3772
		2	33.4600	39.3314	54.3044	28.2369	33.2487	46.0637	20.0043	23.6350	32.9684
		5	35.3137	41.4795	57.1846	29.7823	35.0438	48.4819	21.0709	24.8797	34.6606
	0	0.5	44.8383	52.8299	73.3020	38.0865	44.9520	62.5870	27.3375	32.3762	45.3895
		1	45.2406	53.2976	73.9349	38.4248	45.3462	63.1230	27.5750	32.6543	45.7712
		2	46.0317	54.2178	75.1817	39.0899	46.1218	64.1787	28.0417	33.2011	46.5226
		5	48.3069	56.8680	78.7802	41.0017	48.3540	67.2244	29.3818	34.7731	48.6882
	0.5	0.5	56.0522	66.1331	92.0273	47.7837	56.4759	78.8643	34.5455	40.9720	57.6173
		1	56.5277	66.6883	92.7847	48.1855	56.9460	79.5089	34.8303	41.3069	58.0809
		2	57.4643	67.7822	94.2784	48.9767	57.8724	80.7800	35.3908	41.9663	58.9945
		5	60.1678	70.9431	98.6029	51.2594	60.5480	84.4587	37.0063	43.8690	61.6367
0.5	-0.5	0.5	47.9871	56.5187	78.3528	40.7206	48.0422	66.8298	29.1693	34.5311	48.3640
		1	48.4295	57.0322	79.0459	41.0922	48.4746	67.4160	29.4297	34.8354	48.7804
		2	49.2991	58.0420	80.4102	41.8226	49.3247	68.5699	29.9411	35.4334	49.5997
		5	51.7965	60.9462	84.3431	43.9188	51.7680	71.8945	31.4072	37.1503	51.9578
	0	0.5	59.3998	70.0503	97.3848	50.5863	59.7597	83.3671	36.4973	43.2647	60.7766
		1	59.9148	70.6509	98.2027	51.0210	60.2677	84.0623	36.8046	43.6256	61.2751
		2	60.9288	71.8336	99.8149	51.8765	61.2680	85.4324	37.4092	44.3359	62.2573
		5	63.8520	75.2472	104.4770	54.3417	64.1538	89.3929	39.1495	46.3828	65.0940
	0.5	0.5	70.2257	82.9017	115.4980	59.9550	70.9007	99.1251	43.4722	51.5884	72.6345
		1	70.8150	83.5909	116.4417	60.4540	71.4856	99.9300	43.8273	52.0067	73.2157
		2	71.9762	84.9495	118.3034	61.4370	72.6385	101.5176	44.5266	52.8309	74.3618
		5	75.3310	88.8784	123.6968	64.2759	75.9714	106.1157	46.5443	55.2113	77.6786

Table 6.2

Values of frequency parameter Ω for C-C plate vibrating in second mode for $\varepsilon = 0.3$

			η								
			-0.5			0.0			1.0		
			μ			μ			μ		
α	β	p	-0.5	0.0	1.0	-0.5	0.0	1.0	-0.5	0.0	1.0
-0.5	0	0.5	82.4282	97.1340	134.5806	69.6988	82.2570	114.3119	49.6104	58.7242	82.0998
		1	82.8038	97.5741	135.1833	70.0158	82.6289	114.8228	49.8348	58.9884	82.4649
		2	83.5482	98.4465	136.3788	70.6437	83.3660	115.8359	50.2792	59.5116	83.1886
		5	85.7291	101.0045	139.8885	72.4822	85.5257	118.8086	51.5784	61.0427	85.3096
	0.5	0.5	114.2804	134.4830	185.8168	97.1655	114.5132	158.6964	69.9279	82.6587	115.2368
		1	114.7703	135.0557	186.5973	97.5812	114.9999	159.3617	70.2255	83.0083	115.7177
		2	115.7425	136.1924	188.1470	98.4058	115.9656	160.6825	70.8158	83.7017	116.6722
		5	118.5997	139.5350	192.7088	100.8283	118.8042	164.5690	72.5477	85.7378	119.4784
0	-0.5	0.5	91.8406	108.2468	150.0357	77.5899	91.5889	127.3329	55.1282	65.2705	91.2936
		1	92.2704	108.7509	150.7274	77.9522	92.0144	127.9184	55.3841	65.5721	91.7111
		2	93.1221	109.7499	152.0986	78.6698	92.8574	129.0792	55.8908	66.1691	92.5384
		5	95.6153	112.6766	156.1213	80.7692	95.3257	132.4827	57.3709	67.9149	94.9613
	0	0.5	124.6547	146.7286	202.8367	105.8822	124.8192	173.0684	76.0488	89.9194	125.4297
		1	125.2016	147.3683	203.7097	106.3456	125.3621	173.8117	76.3799	90.3086	125.9659
		2	126.2864	148.6376	205.4424	107.2648	126.4393	175.2870	77.0361	91.0802	127.0296
		5	129.4722	152.3673	210.5397	109.9626	129.6030	179.6250	78.9603	93.3441	130.1547
	0.5	0.5	154.0108	181.1376	250.0011	131.2290	154.5741	213.9764	94.8494	112.0579	156.0527
		1	154.6659	181.9028	251.0422	131.7859	155.2258	214.8658	95.2498	112.5280	156.6985
		2	155.9661	183.4218	253.1097	132.8912	156.5192	216.6319	96.0443	113.4608	157.9805
		5	159.7903	187.8921	259.1996	136.1407	160.3240	221.8323	98.3779	116.2025	161.7530
0.5	-0.5	0.5	134.8044	158.7098	219.4934	114.4054	134.8970	187.1255	82.0275	97.0117	135.3878
		1	135.4076	159.4160	220.4582	114.9162	135.4958	187.9462	82.3916	97.4400	135.9786
		2	136.6041	160.8167	222.3729	115.9288	136.6834	189.5748	83.1132	98.2889	137.1505
		5	140.1153	164.9302	228.0017	118.8992	140.1690	194.3605	85.2272	100.7779	140.5913
	0	0.5	164.8247	193.9009	267.7382	140.3215	165.3237	228.9652	101.2428	119.6415	166.6980
		1	165.5379	194.7346	268.8736	140.9273	166.0329	229.9343	101.6775	120.1522	167.4004
		2	166.9534	196.3892	271.1281	142.1293	167.4404	231.8583	102.5398	121.1652	168.7944
		5	171.1140	201.2557	277.7652	145.6611	171.5783	237.5206	105.0708	124.1407	172.8940
	0.5	0.5	193.0305	226.9549	313.0252	164.6913	193.9257	268.2701	119.3442	140.9521	196.1613
		1	193.8493	227.9109	314.3244	165.3884	194.7409	269.3815	119.8468	141.5418	196.9706
		2	195.4747	229.8090	316.9048	166.7720	196.3593	271.5888	120.8440	142.7121	198.5775
		5	200.2573	235.3966	324.5079	170.8414	201.1217	278.0906	123.7745	146.1535	203.3078

Table 6.3
Values of frequency parameter Ω for C-C plate vibrating in third mode for ε = 0.3

			η								
			-0.5			0.0			1.0		
			μ			μ			μ		
α	β	p	-0.5	0.0	1.0	-0.5	0.0	1.0	-0.5	0.0	1.0
-0.5	0	0.5	162.4630	191.5898	265.6516	137.4146	162.2789	225.6464	97.8960	115.9340	162.1148
		1	162.8812	192.0798	266.3226	137.7694	162.6951	226.2176	98.1500	116.2328	162.5270
		2	163.7133	193.0549	267.6582	138.4749	163.5230	227.3545	98.6550	116.8269	163.3471
		5	166.1751	195.9413	271.6153	140.5615	165.9725	230.7215	100.1467	118.5829	165.7738
	0.5	0.5	223.7253	263.0834	362.7285	190.2089	223.9821	309.6757	136.9143	161.6736	224.7748
		1	224.2744	263.7248	363.6012	190.6768	224.5293	310.4221	137.2525	162.0701	225.3184
		2	225.3677	265.0021	365.3394	191.6081	225.6186	311.9084	137.9253	162.8592	226.4007
		5	228.6084	268.7895	370.4974	194.3676	228.8479	316.3178	139.9173	165.1967	229.6094
0	-0.5	0.5	182.1707	214.9232	298.2455	153.9591	181.8988	253.1417	109.4998	129.7389	181.5862
		1	182.6486	215.4837	299.0142	154.3641	182.3743	253.7956	109.7892	130.0796	182.0572
		2	183.5992	216.5987	300.5440	155.1695	183.3201	255.0967	110.3646	130.7572	182.9943
		5	186.4108	219.8981	305.0751	157.5502	186.1176	258.9487	112.0634	132.7589	185.7657
	0	0.5	245.3629	288.6722	398.3897	208.4147	245.5472	339.8272	149.7401	176.9150	246.2234
		1	245.9753	289.3880	399.3650	208.9360	246.1573	340.6606	150.1161	177.3563	246.8294
		2	247.1942	290.8132	401.3074	209.9734	247.3718	342.3201	150.8643	178.2344	248.0358
		5	250.8062	295.0377	407.0693	213.0462	250.9706	347.2416	153.0782	180.8345	251.6109
	0.5	0.5	301.6038	354.2611	487.3299	256.9406	302.2215	416.9004	185.6962	219.0330	303.8355
		1	302.3385	355.1183	488.4935	257.5676	302.9540	417.8973	186.1510	219.5657	304.5642
		2	303.8014	356.8255	490.8114	258.8160	304.4128	419.8829	187.0562	220.6262	306.0154
		5	308.1403	361.8906	497.6927	262.5173	308.7393	425.7762	189.7379	223.7696	310.3201
0.5	-0.5	0.5	266.5486	313.7337	433.3371	226.2302	266.6560	369.3584	162.2767	191.8171	267.2071
		1	267.2234	314.5232	434.4141	226.8042	267.3284	370.2780	162.6901	192.3026	267.8748
		2	268.5666	316.0946	436.5586	227.9464	268.6665	372.1090	163.5125	193.2687	269.2038
		5	272.5454	320.7516	442.9187	231.3285	272.6305	377.5373	165.9452	196.1278	273.1410
	0	0.5	324.1226	380.8817	524.4009	275.9019	324.6730	448.2677	199.0731	234.9237	326.1819
		1	324.9222	381.8152	525.6695	276.5837	325.4702	449.3538	199.5669	235.5025	326.9747
		2	326.5141	383.6742	528.1965	277.9412	327.0574	451.5170	200.5496	236.6547	328.5534
		5	331.2344	389.1880	535.6965	281.9646	331.7636	457.9355	203.4600	240.0684	333.2347
	0.5	0.5	378.0487	443.7488	609.5913	322.4594	379.0286	522.1341	233.6163	275.3708	381.4652
		1	378.9667	444.8192	611.0422	323.2438	379.9443	523.3785	234.1865	276.0382	382.3769
		2	380.7950	446.9513	613.9326	324.8056	381.7680	525.8572	235.3214	277.3670	384.1926
		5	386.2189	453.2784	622.5152	329.4377	387.1786	533.2158	238.6852	281.3070	389.5800

Table 6.4
Values of frequency parameter Ω for C-C plate vibrating in fundamental mode for ε = 0.5

			η								
			-0.5			0.0			1.0		
			μ			μ			μ		
α	β	p	-0.5	0.0	1.0	-0.5	0.0	1.0	-0.5	0.0	1.0
-0.5	0	0.5	54.8694	66.1664	96.1841	45.3396	54.7228	79.6895	30.8760	37.3315	54.5555
		1	55.0406	66.3711	96.4768	45.4803	54.8912	79.9310	30.9707	37.4451	54.7192
		2	55.3810	66.7782	97.0593	45.7600	55.2262	80.4115	31.1589	37.6711	55.0450
		5	56.3876	67.9824	98.7828	46.5873	56.2171	81.8331	31.7155	38.3394	56.0085
	0.5	0.5	82.4455	99.5391	145.0497	68.3869	82.6392	120.6382	46.9278	56.8089	83.2275
		1	82.6856	99.8274	145.4660	68.5852	82.8778	120.9834	47.0626	56.9715	83.4640
		2	83.1634	100.4013	146.2946	68.9800	83.3525	121.6706	47.3311	57.2951	83.9348
		5	84.5783	102.1010	148.7496	70.1488	84.7584	123.7063	48.1257	58.2534	85.3293
0	-0.5	0.5	60.6863	73.1623	106.2975	50.0943	60.4458	87.9759	34.0431	41.1495	60.1008
		1	60.8822	73.3962	106.6310	50.2552	60.6381	88.2508	34.1511	41.2789	60.2866
		2	61.2718	73.8615	107.2945	50.5750	61.0205	88.7974	34.3658	41.5362	60.6562
		5	62.4231	75.2367	109.2567	51.5199	62.1506	90.4137	34.9999	42.2965	61.7490
	0	0.5	88.8951	107.2912	156.2404	73.6658	88.9892	129.8186	50.4529	61.0555	89.3850
		1	89.1591	107.6079	156.6965	73.8837	89.2508	130.1962	50.6006	61.2334	89.6431
		2	89.6844	108.2380	157.6041	74.3171	89.7713	130.9479	50.8946	61.5873	90.1567
		5	91.2393	110.1035	160.2920	75.5999	91.3123	133.1738	51.7645	62.6349	91.6774
	0.5	0.5	115.4937	139.4880	203.4110	95.9041	115.9318	169.3619	65.9532	79.8685	117.0962
		1	115.8277	139.8897	203.9925	96.1804	116.2646	169.8450	66.1417	80.0961	117.4281
		2	116.4924	140.6892	205.1502	96.7304	116.9269	170.8064	66.5168	80.5490	118.0889
		5	118.4611	143.0575	208.5804	98.3593	118.8889	173.6553	67.6277	81.8903	120.0466
0.5	-0.5	0.5	95.1861	114.8545	167.1626	78.8134	95.1825	138.7762	53.8882	65.1951	95.3902
		1	95.4742	115.1996	167.6585	79.0509	95.4673	139.1864	54.0490	65.3883	95.6699
		2	96.0474	115.8863	168.6453	79.5233	96.0340	140.0027	54.3687	65.7728	96.2264
		5	97.7434	117.9186	171.5668	80.9210	97.7109	142.4193	55.3144	66.9102	97.8737
	0	0.5	122.1732	147.5156	214.9960	101.3733	122.5098	178.8691	69.6080	84.2707	123.4771
		1	122.5308	147.9452	215.6169	101.6689	122.8654	179.3843	69.8092	84.5134	123.8304
		2	123.2424	148.8002	216.8527	102.2571	123.5730	180.4096	70.2096	84.9962	124.5336
		5	125.3494	151.3323	220.5137	103.9986	125.6684	183.4469	71.3947	86.4258	126.6162
	0.5	0.5	148.3643	179.2228	261.4619	123.2743	149.0471	217.8285	84.8786	102.8073	150.7893
		1	148.7924	179.7381	262.2092	123.6288	149.4744	218.4497	85.1208	103.0999	151.2170
		2	149.6443	180.7636	263.6967	124.3344	150.3248	219.6863	85.6029	103.6824	152.0683
		5	152.1679	183.8019	268.1045	126.4242	152.8440	223.3503	87.0306	105.4078	154.5905

Table 6.5

Values of frequency parameter Ω for C-C plate vibrating in second mode for $\varepsilon = 0.5$

			η								
			-0.5			0.0			1.0		
			μ			μ			μ		
α	β	p	-0.5	0.0	1.0	-0.5	0.0	1.0	-0.5	0.0	1.0
-0.5	0	0.5	151.9198	183.3942	266.9443	125.5857	151.7207	221.1814	85.6232	103.6006	151.4958
		1	152.1449	183.6653	267.3373	125.7716	151.9449	221.5067	85.7497	103.7533	151.7182
		2	152.5940	184.2063	268.1213	126.1424	152.3920	222.1558	86.0020	104.0580	152.1618
		5	153.9318	185.8183	270.4580	127.2472	153.7242	224.0900	86.7534	104.9654	153.4837
	0.5	0.5	226.5247	273.2211	397.0149	187.8800	226.7838	330.0419	128.9476	155.8869	227.5607
		1	226.8497	273.6121	397.5796	188.1494	227.1080	330.5110	129.1321	156.1094	227.8836
		2	227.4983	274.3922	398.7065	188.6868	227.7550	331.4471	129.5002	156.5533	228.5280
		5	229.4316	276.7178	402.0663	190.2889	229.6836	334.2380	130.5974	157.8764	230.4489
0	-0.5	0.5	169.3796	204.5168	297.8208	139.8935	169.0444	246.5465	95.2061	115.2222	168.5680
		1	169.6359	204.8257	298.2691	140.1050	169.2995	246.9172	95.3497	115.3957	168.8209
		2	170.1472	205.4420	299.1634	140.5268	169.8083	247.6568	95.6360	115.7417	169.3254
		5	171.6701	207.2779	301.8279	141.7831	171.3241	249.8602	96.4887	116.7721	170.8282
	0	0.5	245.8553	296.6008	431.1730	203.7435	245.9860	358.1435	139.6018	168.8048	246.5283
		1	246.2130	297.0313	431.7954	204.0397	246.3428	358.6601	139.8044	169.0492	246.8832
		2	246.9267	297.8902	433.0373	204.6307	247.0545	359.6908	140.2084	169.5366	247.5914
		5	249.0541	300.4504	436.7395	206.3919	249.1757	362.7634	141.4124	170.9891	249.7024
	0.5	0.5	317.6318	383.0149	556.2786	263.6942	318.2184	462.8786	181.3234	219.1520	319.7568
		1	318.0874	383.5626	557.0690	264.0721	318.6732	463.5359	181.5828	219.4646	320.2102
		2	318.9964	384.6555	558.6463	264.8261	319.5805	464.8475	182.1003	220.0883	321.1148
		5	321.7063	387.9139	563.3493	267.0739	322.2854	468.7582	183.6428	221.9477	323.8119
0.5	-0.5	0.5	264.6915	319.3842	464.4647	219.1954	264.6912	385.5215	149.9718	181.3787	264.9929
		1	265.0816	319.8538	465.1442	219.5181	265.0800	386.0850	150.1921	181.6446	265.3795
		2	265.8598	320.7906	466.4999	220.1619	265.8557	387.2093	150.6315	182.1749	266.1508
		5	268.1788	323.5827	470.5409	222.0803	268.1673	390.5604	151.9408	183.7552	268.4495
	0	0.5	337.6301	407.2003	591.6089	280.1120	338.0902	491.9560	192.3581	232.5302	339.3977
		1	338.1189	407.7882	592.4579	280.5172	338.5780	492.6616	192.6358	232.8650	339.8836
		2	339.0942	408.9612	594.1519	281.3256	339.5511	494.0693	193.1899	233.5330	340.8532
		5	342.0013	412.4580	599.2028	283.7353	342.4518	498.2662	194.8410	235.5241	343.7437
	0.5	0.5	408.2250	492.1885	714.6398	339.0836	409.1400	594.9673	233.4100	282.0671	411.4408
		1	408.8108	492.8926	715.6553	339.5698	409.7249	595.8123	233.7441	282.4696	412.0243
		2	409.9796	494.2974	717.6818	340.5398	410.8918	597.4983	234.4106	283.2728	413.1886
		5	413.4642	498.4861	723.7243	343.4318	414.3710	602.5257	236.3974	285.6670	416.6599

Table 6.6
Values of frequency parameter Ω for C-C plate vibrating in third mode for $\varepsilon = 0.5$

			η								
			-0.5			0.0			1.0		
			μ			μ			μ		
α	β	p	-0.5	0.0	1.0	-0.5	0.0	1.0	-0.5	0.0	1.0
-0.5	0	0.5	298.4179	360.4052	524.8828	246.7351	298.2008	434.9154	168.3098	203.7088	297.9567
		1	298.6657	360.7037	525.3153	246.9403	298.4481	435.2742	168.4502	203.8782	298.2031
		2	299.1606	361.2999	526.1792	247.3500	298.9421	435.9908	168.7305	204.2165	298.6952
		5	300.6397	363.0817	528.7614	248.5746	300.4182	438.1327	169.5681	205.2275	300.1659
	0.5	0.5	443.4895	534.6896	776.0420	367.8180	443.7733	645.0058	252.4654	305.0352	444.6226
		1	443.8486	535.1213	776.6651	368.1161	444.1320	645.5241	252.6705	305.2823	444.9805
		2	444.5659	535.9837	777.9097	368.7116	444.8484	646.5593	253.0801	305.7757	445.6953
		5	446.7102	538.5618	781.6309	370.4917	446.9900	649.6544	254.3044	307.2505	447.8322
0	-0.5	0.5	333.8734	403.4106	588.0437	275.8195	333.5068	486.8520	187.8332	227.4462	332.9877
		1	334.1550	403.7500	588.5360	276.0525	333.7878	487.2601	187.9923	227.6383	333.2676
		2	334.7172	404.4277	589.5193	276.5176	334.3488	488.0751	188.3100	228.0220	333.8264
		5	336.3974	406.4531	592.4581	277.9075	336.0254	490.5109	189.2592	229.1684	335.4963
	0	0.5	482.6849	582.1954	845.7079	400.0157	482.8294	702.3733	274.1382	331.3669	483.4254
		1	483.0798	582.6704	846.3941	400.3433	483.2237	702.9437	274.3633	331.6381	483.8187
		2	483.8686	583.6192	847.7649	400.9977	484.0113	704.0831	274.8128	332.1798	484.6044
		5	486.2264	586.4556	851.8631	402.9537	486.3658	707.4895	276.1561	333.7990	486.9530
	0.5	0.5	622.1005	749.6594	1086.9667	516.4101	622.7426	904.2330	355.0838	428.8112	624.4232
		1	622.6037	750.2640	1087.8383	516.8282	623.2453	904.9586	355.3718	429.1580	624.9250
		2	623.6087	751.4717	1089.5794	517.6632	624.2494	906.4079	355.9471	429.8506	625.9273
		5	626.6136	755.0826	1094.7852	520.1597	627.2512	910.7411	357.6668	431.9211	628.9239
0.5	-0.5	0.5	520.8915	628.5124	913.6587	431.3888	520.8929	758.3063	295.2396	357.0094	521.2281
		1	521.3217	629.0301	914.4073	431.7455	521.3225	758.9282	295.4843	357.3045	521.6564
		2	522.1808	630.0641	915.9025	432.4578	522.1803	760.1703	295.9730	357.8938	522.5119
		5	524.7489	633.1549	920.3726	434.5869	524.7444	763.8834	297.4335	359.6549	525.0690
	0	0.5	662.6301	798.7724	1158.9604	549.7164	663.1353	963.5402	377.5190	456.0639	664.5679
		1	663.1699	799.4212	1159.8965	550.1646	663.6744	964.3190	377.8274	456.4354	665.1059
		2	664.2479	800.7171	1161.7662	551.0597	664.7511	965.8746	378.4435	457.1773	666.1805
		5	667.4706	804.5914	1167.3565	553.7357	667.9702	970.5253	380.2847	459.3952	669.3932
	0.5	0.5	799.6828	963.3871	1396.0836	664.1517	800.6840	1161.9612	457.1229	551.8858	803.1970
		1	800.3295	964.1640	1397.2028	664.6893	801.3302	1162.8933	457.4936	552.3319	803.8421
		2	801.6214	965.7158	1399.4385	665.7631	802.6209	1164.7552	458.2340	553.2230	805.1309
		5	805.4837	970.3556	1406.1235	668.9734	806.4797	1170.3221	460.4474	555.8870	808.9840

Table 6.7

Values of frequency parameter Ω for C-S plate vibrating in fundamental mode for ε = 0.3

			η								
			-0.5			0.0			1.0		
			μ			μ			μ		
α	β	p	-0.5	0.0	1.0	-0.5	0.0	1.0	-0.5	0.0	1.0
-0.5	0	0.5	21.5004	25.0522	33.9253	17.9791	20.9796	28.4916	12.5029	14.6309	19.9814
		1	21.7966	25.4001	34.4067	18.2248	21.2689	28.8937	12.6709	14.8296	20.2600
		2	22.3751	26.0800	35.3479	18.7046	21.8341	29.6797	12.9988	15.2176	20.8044
		5	24.0125	28.0065	38.0189	20.0618	23.4346	31.9092	13.9249	16.3146	22.3466
	0.5	0.5	27.1855	31.5546	42.3620	22.8132	26.5148	35.6883	15.9747	18.6149	25.1827
		1	27.5965	32.0478	43.0805	23.1568	26.9280	36.2929	16.2132	18.9029	25.6078
		2	28.3998	33.0112	44.4821	23.8283	27.7350	37.4724	16.6791	19.4653	26.4369
		5	30.6773	35.7406	48.4409	25.7311	30.0208	40.8031	17.9982	21.0571	28.7774
0	-0.5	0.5	24.3079	28.3615	38.5134	20.3246	23.7489	32.3437	14.1308	16.5591	22.6808
		1	24.6425	28.7521	39.0466	20.6020	24.0735	32.7887	14.3202	16.7817	22.9887
		2	25.2956	29.5150	40.0891	21.1434	24.7073	33.6587	14.6897	17.2161	23.5903
		5	27.1423	31.6753	43.0474	22.6730	26.5007	36.1261	15.7320	18.4437	25.2943
	0	0.5	30.2542	35.1598	47.3170	25.3828	29.5386	39.8578	17.7661	20.7298	28.1177
		1	30.6950	35.6845	48.0689	25.7509	29.9777	40.4898	18.0210	21.0352	28.5611
		2	31.5566	36.7102	49.5375	26.4703	30.8359	41.7242	18.5190	21.6319	29.4269
		5	34.0009	39.6192	53.6962	28.5100	33.2692	45.2188	19.9296	23.3222	31.8766
	0.5	0.5	35.3057	40.9189	54.7514	29.6774	34.4397	46.1963	20.8496	24.2558	32.6948
		1	35.8742	41.6072	55.7757	30.1541	35.0181	47.0608	21.1826	24.6614	33.3061
		2	36.9842	42.9500	57.7694	31.0849	36.1465	48.7434	21.8325	25.4527	34.4958
		5	40.1252	46.7435	63.3709	33.7181	39.3339	53.4709	23.6698	27.6866	37.8381
0.5	-0.5	0.5	33.2664	38.7028	52.1990	27.9037	32.5085	43.9636	19.5216	22.8047	31.0049
		1	33.7394	39.2623	52.9898	28.2983	32.9763	44.6278	19.7943	23.1294	31.4699
		2	34.6640	40.3562	54.5357	29.0695	33.8908	45.9258	20.3271	23.7640	32.3786
		5	37.2871	43.4603	58.9199	31.2563	36.4845	49.6061	21.8363	25.5621	34.9529
	0	0.5	38.4878	44.6531	59.8682	32.3440	37.5741	50.5053	22.7116	26.4515	35.7327
		1	39.0813	45.3668	60.9152	32.8413	38.1733	51.3880	23.0582	26.8709	36.3557
		2	40.2410	46.7604	62.9562	33.8127	39.3433	53.1089	23.7350	27.6896	37.5701
		5	43.5271	50.7054	68.7111	36.5644	42.6543	57.9607	25.6509	30.0051	40.9931
	0.5	0.5	43.2479	50.0729	66.8524	36.3904	42.1856	56.4582	25.6163	29.7681	40.0289
		1	43.9772	50.9613	68.1931	37.0030	42.9335	57.5915	26.0457	30.2945	40.8329
		2	45.3998	52.6926	70.7974	38.1982	44.3911	59.7931	26.8832	31.3202	42.3946
		5	49.4179	57.5707	78.0834	41.5732	48.4975	65.9532	29.2471	34.2089	46.7648

Table 6.8
Values of frequency parameter Ω for C-S plate vibrating in second mode for $\varepsilon = 0.3$

			η								
			-0.5			0.0			1.0		
			μ			μ			μ		
α	β	p	-0.5	0.0	1.0	-0.5	0.0	1.0	-0.5	0.0	1.0
-0.5	0	0.5	67.9479	79.8483	109.9679	57.2550	67.3823	93.0767	40.4809	47.7821	66.3970
		1	68.3245	80.2919	110.5832	57.5716	67.7557	93.5959	40.7035	48.0453	66.7647
		2	69.0700	81.1702	111.8019	58.1980	68.4948	94.6238	41.1435	48.5658	67.4924
		5	71.2466	83.7366	115.3668	60.0259	70.6527	97.6293	42.4254	50.0834	69.6174
	0.5	0.5	91.1061	106.7586	146.1515	77.2107	90.6122	124.4234	55.2267	65.0091	89.8131
		1	91.6160	107.3603	146.9902	77.6408	91.1204	125.1332	55.5314	65.3699	90.3191
		2	92.6260	108.5525	148.6522	78.4927	92.1272	126.5395	56.1345	66.0843	91.3215
		5	95.5822	112.0436	153.5225	80.9851	95.0739	130.6592	57.8970	68.1733	94.2556
0	-0.5	0.5	76.6451	90.1295	124.3051	64.5258	75.9911	105.1205	45.5360	53.7864	74.8505
		1	77.0726	90.6331	125.0034	64.8848	76.4146	105.7092	45.7879	54.0843	75.2669
		2	77.9184	91.6298	126.3861	65.5950	77.2526	106.8747	46.2859	54.6735	76.0907
		5	80.3869	94.5404	130.4289	67.6661	79.6979	110.2805	47.7359	56.3902	78.4950
	0	0.5	100.4407	117.7873	161.5124	85.0281	99.8632	137.3503	60.6804	71.4833	98.9181
		1	101.0028	118.4502	162.4348	85.5020	100.4228	138.1307	61.0156	71.8801	99.4741
		2	102.1160	119.7635	164.2625	86.4404	101.5312	139.6769	61.6791	72.6658	100.5754
		5	105.3727	123.6074	169.6165	89.1843	104.7739	144.2043	63.6168	74.9618	103.7973
	0.5	0.5	121.7900	142.5843	194.8187	103.4474	121.2953	166.2406	74.3272	87.4199	120.5589
		1	122.4808	143.4003	195.9585	104.0311	121.9854	167.2063	74.7417	87.9111	121.2491
		2	123.8496	145.0175	198.2175	105.1872	123.3528	169.1199	75.5626	88.8843	122.6166
		5	127.8584	149.7551	204.8387	108.5719	127.3573	174.7278	77.9636	91.7318	126.6216
0.5	-0.5	0.5	109.6256	128.6398	176.6314	92.7173	108.9630	150.0685	66.0409	77.8471	107.8691
		1	110.2397	129.3639	177.6378	93.2348	109.5739	150.9197	66.4065	78.2798	108.4751
		2	111.4559	130.7982	179.6319	94.2594	110.7838	152.6059	67.1300	79.1363	109.6751
		5	115.0123	134.9946	185.4712	97.2538	114.3217	157.5418	69.2417	81.6381	113.1842
	0	0.5	131.4195	153.9569	210.6459	111.5174	130.8413	179.5692	79.9644	94.1092	129.9592
		1	132.1634	154.8352	211.8704	112.1456	131.5838	180.6065	80.4101	94.6373	130.7003
		2	133.6372	156.5753	214.2973	113.3898	133.0546	182.6621	81.2927	95.6831	132.1686
		5	137.9519	161.6718	221.4089	117.0309	137.3606	188.6842	83.8726	98.7420	136.4671
	0.5	0.5	151.9354	177.7809	242.6293	129.2292	151.4458	207.3300	93.1051	109.4519	150.7837
		1	152.8064	178.8102	244.0686	129.9655	152.3169	208.5503	93.6291	110.0732	151.6573
		2	154.5321	180.8501	246.9214	131.4244	154.0432	210.9687	94.6668	111.3038	153.3882
		5	159.5873	186.8272	255.2838	135.6964	159.0997	218.0567	97.7030	114.9061	158.4587

Table 6.9
Values of frequency parameter Ω for C-S plate vibrating in third mode for ε = 0.3

			η								
			-0.5			0.0			1.0		
			μ			μ			μ		
α	β	p	-0.5	0.0	1.0	-0.5	0.0	1.0	-0.5	0.0	1.0
-0.5	0	0.5	141.3116	166.4334	230.1292	119.3169	140.7255	195.1316	84.7248	100.2065	139.7321
		1	141.7322	166.9279	230.8120	119.6727	141.1443	195.7110	84.9784	100.5056	140.1478
		2	142.5683	167.9113	232.1702	120.3799	141.9770	196.8635	85.4821	101.1000	140.9743
		5	145.0381	170.8174	236.1874	122.4677	144.4365	200.2708	86.9674	102.8538	143.4152
	0.5	0.5	191.3759	224.5814	308.2485	162.4646	190.9223	262.7903	116.6238	137.4396	190.2510
		1	191.9412	225.2457	309.1649	162.9444	191.4866	263.5701	116.9681	137.8453	190.8137
		2	193.0658	226.5673	310.9885	163.8987	192.6093	265.1218	117.6526	138.6522	191.9332
		5	196.3931	230.4790	316.3888	166.7210	195.9306	269.7155	119.6752	141.0374	195.2454
0	-0.5	0.5	159.5621	188.0645	260.4136	134.6054	158.8747	220.6220	95.4034	112.9227	157.7051
		1	160.0398	188.6265	261.1899	135.0093	159.3503	221.2805	95.6908	113.2620	158.1770
		2	160.9895	189.7439	262.7340	135.8121	160.2959	222.5901	96.2618	113.9361	159.1151
		5	163.7940	193.0452	267.3003	138.1813	163.0881	226.4611	97.9447	115.9245	161.8852
	0	0.5	211.0532	247.8681	340.7512	178.9859	210.5050	290.2100	128.2152	151.2220	209.6718
		1	211.6788	248.6033	341.7653	179.5167	211.1294	291.0727	128.5956	151.6705	210.2940
		2	212.9234	250.0661	343.7831	180.5723	212.3713	292.7893	129.3520	152.5623	211.5317
		5	216.6047	254.3943	349.7575	183.6933	216.0447	297.8700	131.5859	155.1976	215.1924
	0.5	0.5	257.0764	301.2878	412.4244	218.6952	256.6706	352.3502	157.6431	185.5462	256.1756
		1	257.8388	302.1835	413.6597	219.3431	257.4324	353.4025	158.1092	186.0954	256.9369
		2	259.3559	303.9660	416.1182	220.6321	258.9483	355.4967	159.0363	187.1879	258.4516
		5	263.8464	309.2437	423.4010	224.4463	263.4353	361.6989	161.7775	190.4192	262.9349
0.5	-0.5	0.5	230.4010	270.7727	372.7432	195.2219	229.7562	317.1846	139.5942	164.7566	228.7570
		1	231.0865	271.5783	373.8542	195.8031	230.4400	318.1296	140.0103	165.2473	229.4380
		2	232.4498	273.1809	376.0650	196.9589	231.8001	320.0097	140.8375	166.2230	230.7926
		5	236.4814	277.9218	382.6094	200.3752	235.8218	325.5733	143.2800	169.1052	234.7981
	0	0.5	277.4320	325.3632	445.9880	235.7999	276.9330	380.6885	169.6623	199.8296	276.2796
		1	278.2564	326.3317	447.3232	236.5002	277.7565	381.8257	170.1657	200.4227	277.1019
		2	279.8967	328.2589	449.9805	237.8933	279.3949	384.0889	171.1667	201.6025	278.7380
		5	284.7508	333.9640	457.8509	242.0145	284.2435	390.7903	174.1256	205.0913	283.5795
	0.5	0.5	321.5853	376.5951	514.6790	273.9178	321.2329	440.2749	197.9462	232.8071	320.9248
		1	322.5429	377.7199	516.2297	274.7322	322.1903	441.5969	198.5331	233.4984	321.8825
		2	324.4485	379.9585	519.3165	276.3526	324.0955	444.2280	199.7005	234.8736	323.7883
		5	330.0904	386.5880	528.4617	281.1486	329.7361	452.0218	203.1533	238.9427	329.4309

Table 6.10

Values of frequency parameter Ω for C-S plate vibrating in fundamental mode for $\varepsilon = 0.5$

			η = -0.5			η = 0.0			η = 1.0		
			μ			μ			μ		
α	β	p	-0.5	0.0	1.0	-0.5	0.0	1.0	-0.5	0.0	1.0
-0.5	0	0.5	39.5065	47.3089	67.7484	32.3930	38.8193	55.6729	21.7157	26.0619	37.4855
		1	39.6941	47.5364	68.0839	32.5465	39.0056	55.9483	21.8180	26.1864	37.6703
		2	40.0667	47.9880	68.7498	32.8512	39.3754	56.4948	22.0211	26.4334	38.0371
		5	41.1630	49.3169	70.7085	33.7477	40.4635	58.1024	22.6185	27.1602	39.1159
	0.5	0.5	54.6253	65.2706	93.0162	44.8926	53.6783	76.6003	30.2321	36.1979	51.7942
		1	54.9213	65.6351	93.5726	45.1359	53.9781	77.0590	30.3957	36.4000	52.1048
		2	55.5086	66.3578	94.6752	45.6184	54.5726	77.9681	30.7203	36.8009	52.7204
		5	57.2334	68.4795	97.9069	47.0357	56.3180	80.6326	31.6737	37.9776	54.5248
0	-0.5	0.5	44.9200	53.8326	77.2195	36.8150	44.1529	63.4298	24.6579	29.6168	42.6734
		1	45.1292	54.0850	77.5879	36.9859	44.3594	63.7318	24.7715	29.7544	42.8756
		2	45.5445	54.5861	78.3191	37.3252	44.7693	64.3314	24.9972	30.0276	43.2771
		5	46.7665	56.0608	80.4711	38.3234	45.9753	66.0955	25.6609	30.8314	44.4583
	0	0.5	60.5342	72.3876	103.3284	49.7268	59.5060	85.0591	33.4584	40.0940	57.4690
		1	60.8462	72.7697	103.9049	49.9829	59.8199	85.5340	33.6303	40.3052	57.7899
		2	61.4654	73.5277	105.0483	50.4912	60.4429	86.4757	33.9715	40.7243	58.4263
		5	63.2857	75.7552	108.4045	51.9852	62.2733	89.2399	34.9743	41.9557	60.2944
	0.5	0.5	74.9352	89.4814	127.3385	61.6305	73.6438	104.9398	41.5658	49.7342	71.0556
		1	75.3621	90.0093	128.1524	61.9818	74.0786	105.6116	41.8027	50.0281	71.5116
		2	76.2086	91.0558	129.7643	62.6782	74.9406	106.9422	42.2725	50.6108	72.4148
		5	78.6923	94.1239	134.4805	64.7219	77.4679	110.8356	43.6509	52.3191	75.0580
0.5	-0.5	0.5	66.3130	79.3503	113.4258	54.4526	65.2049	93.3385	36.6097	43.9009	63.0189
		1	66.6429	79.7525	114.0269	54.7231	65.5351	93.8331	36.7910	44.1226	63.3525
		2	67.2978	80.5506	115.2194	55.2601	66.1904	94.8143	37.1508	44.5626	64.0144
		5	69.2238	82.8975	118.7232	56.8395	68.1171	97.6975	38.2088	45.8564	65.9592
	0	0.5	81.0161	96.8049	137.9446	66.6077	79.6433	113.6436	44.8905	53.7487	76.9000
		1	81.4569	97.3475	138.7734	66.9700	80.0898	114.3272	45.1345	54.0501	77.3633
		2	82.3310	98.4235	140.4159	67.6887	80.9755	115.6821	45.6184	54.6478	78.2816
		5	84.8985	101.5818	145.2291	69.7994	83.5749	119.6523	47.0397	56.4020	80.9727
	0.5	0.5	95.1165	113.5362	161.4328	78.2619	93.4799	133.0897	52.8265	63.1818	90.1866
		1	95.6754	114.2292	162.5071	78.7220	94.0511	133.9771	53.1374	63.5684	90.7897
		2	96.7833	115.6024	164.6341	79.6343	95.1830	135.7340	53.7535	64.3345	91.9838
		5	100.0317	119.6249	170.8500	82.3091	98.4989	140.8690	55.5603	66.5792	95.4745

Table 6.11
Values of frequency parameter Ω for C-S plate vibrating in second mode for ε = 0.5

			η								
			-0.5			0.0			1.0		
			μ			μ			μ		
α	β	p	-0.5	0.0	1.0	-0.5	0.0	1.0	-0.5	0.0	1.0
-0.5	0	0.5	124.8076	150.3508	217.8814	102.9156	124.0725	180.0744	69.8268	84.3093	122.7373
		1	125.0411	150.6337	218.2967	103.1078	124.3055	180.4168	69.9567	84.4669	122.9695
		2	125.5067	151.1979	219.1249	103.4909	124.7700	181.0996	70.2155	84.7811	123.4324
		5	126.8921	152.8766	221.5896	104.6309	126.1523	183.1313	70.9855	85.7159	124.8098
	0.5	0.5	181.4197	218.1887	315.1087	150.1120	180.6750	261.3351	102.5549	123.6264	179.3746
		1	181.7712	218.6153	315.7372	150.4019	181.0269	261.8540	102.7516	123.8654	179.7277
		2	182.4721	219.4658	316.9902	150.9798	181.7286	262.8887	103.1436	124.3419	180.4318
		5	184.5581	221.9973	320.7196	152.6997	183.8168	265.9682	104.3102	125.7598	182.5270
0	-0.5	0.5	140.6413	169.5147	245.9214	115.8659	139.7591	203.0638	78.4672	94.7921	138.1495
		1	140.9040	169.8328	246.3880	116.0819	140.0209	203.4483	78.6130	94.9689	138.4099
		2	141.4275	170.4670	247.3184	116.5125	140.5429	204.2148	78.9035	95.3215	138.9291
		5	142.9853	172.3542	250.0869	117.7935	142.0956	206.4957	79.7676	96.3703	140.4736
	0	0.5	198.5647	238.9271	345.4159	164.1527	197.6720	286.2149	111.9461	135.0130	196.0955
		1	198.9461	239.3897	346.0964	164.4670	198.0535	286.7766	112.1592	135.2719	196.4776
		2	199.7065	240.3119	347.4533	165.0937	198.8140	287.8968	112.5839	135.7879	197.2393
		5	201.9696	243.0568	351.4916	166.9589	201.0775	291.2305	113.8479	137.3236	199.5061
	0.5	0.5	253.1007	304.2736	439.0599	209.6288	252.2083	364.4984	143.4980	172.9149	250.6864
		1	253.5987	304.8783	439.9515	210.0397	252.7074	365.2350	143.7771	173.2542	251.1881
		2	254.5918	306.0839	441.7294	210.8590	253.7025	366.7037	144.3335	173.9308	252.1885
		5	257.5475	309.6721	447.0202	213.2974	256.6641	371.0747	145.9894	175.9441	255.1656
0.5	-0.5	0.5	215.3680	259.2552	375.1324	177.9089	214.3272	310.6015	121.1411	146.1634	212.4743
		1	215.7791	259.7536	375.8649	178.2476	214.7381	311.2060	121.3705	146.4420	212.8851
		2	216.5988	260.7473	377.3254	178.9229	215.5572	312.4114	121.8278	146.9973	213.7043
		5	219.0380	263.7046	381.6721	180.9323	217.9949	315.9985	123.1883	148.6498	216.1418
	0	0.5	270.7234	325.5849	470.1900	224.0661	269.6814	390.0632	153.1615	184.6289	267.8802
		1	271.2516	326.2259	471.1341	224.5017	270.2104	390.8431	153.4572	184.9883	268.4112
		2	272.3049	327.5040	473.0166	225.3704	271.2650	392.3980	154.0468	185.7049	269.4699
		5	275.4395	331.3077	478.6190	227.9556	274.4037	397.0257	155.8011	187.8373	272.6204
	0.5	0.5	324.3913	389.8899	562.3369	268.8229	323.3539	467.1030	184.2216	221.9393	321.6161
		1	325.0357	390.6724	563.4914	269.3548	324.0000	468.0570	184.5831	222.3788	322.2663
		2	326.3206	392.2327	565.7933	270.4152	325.2883	469.9592	185.3038	223.2552	323.5626
		5	330.1447	396.8761	572.6433	273.5710	329.1221	475.6199	187.4483	225.8632	327.4205

Table 6.12
Values of frequency parameter Ω for C-S plate vibrating in third mode for ε = 0.5

			η								
			-0.5			0.0			1.0		
			μ			μ			μ		
α	β	p	-0.5	0.0	1.0	-0.5	0.0	1.0	-0.5	0.0	1.0
-0.5	0	0.5	259.1024	312.6159	454.3342	213.9644	258.3402	375.9932	145.6110	176.0621	256.9814
		1	259.3575	312.9243	454.7849	214.1751	258.5950	376.3659	145.7544	176.2358	257.2359
		2	259.8667	313.5402	455.6847	214.5957	259.1039	377.1101	146.0408	176.5826	257.7441
		5	261.3879	315.3797	458.3728	215.8519	260.6238	379.3333	146.8959	177.6182	259.2620
	0.5	0.5	380.2459	457.8014	662.4667	315.0175	379.5422	550.0139	215.7725	260.3448	378.3727
		1	380.6252	458.2599	663.1366	315.3313	379.9217	550.5688	215.9868	260.6043	378.7528
		2	381.3826	459.1756	664.4744	315.9578	380.6795	551.6768	216.4149	261.1225	379.5116
		5	383.6452	461.9111	668.4713	317.8295	382.9433	554.9869	217.6936	262.6704	381.7787
0	-0.5	0.5	291.5590	351.9992	512.2235	240.5531	290.6287	423.5289	163.4123	197.7131	288.9568
		1	291.8464	352.3468	512.7314	240.7904	290.9158	423.9489	163.5737	197.9086	289.2433
		2	292.4203	353.0408	513.7456	241.2642	291.4890	424.7874	163.8959	198.2989	289.8154
		5	294.1343	355.1137	516.7752	242.6789	293.2009	427.2919	164.8579	199.4642	291.5239
	0	0.5	415.6147	500.6746	725.3570	344.0340	414.7431	601.7319	235.2544	284.0162	413.2618
		1	416.0283	501.1745	726.0872	344.3759	415.1567	602.3365	235.4879	284.2988	413.6757
		2	416.8540	502.1727	727.5452	345.0588	415.9825	603.5438	235.9540	284.8632	414.5022
		5	419.3206	505.1547	731.9012	347.0985	418.4496	607.1505	237.3465	286.5489	416.9710
	0.5	0.5	532.1729	640.3494	925.5469	441.2838	531.3697	769.1444	302.8091	365.1564	530.0958
		1	532.7079	640.9961	926.4917	441.7266	531.9052	769.9272	303.1119	365.5229	530.6324
		2	533.7762	642.2875	928.3783	442.6108	532.9745	771.4903	303.7167	366.2548	531.7040
		5	536.9677	646.1457	934.0147	445.2522	536.1689	776.1604	305.5231	368.4412	534.9053
0.5	-0.5	0.5	450.2218	542.6358	786.9434	372.4142	449.1814	652.3572	254.2941	307.1566	447.3867
		1	450.6690	543.1765	787.7330	372.7839	449.6286	653.0109	254.5464	307.4620	447.8340
		2	451.5621	544.2562	789.3098	373.5222	450.5216	654.3163	255.0501	308.0718	448.7271
		5	454.2299	547.4814	794.0206	375.7276	453.1890	658.2159	256.5545	309.8932	451.3949
	0	0.5	568.6078	684.5018	990.2750	471.1869	567.6355	822.3956	322.9018	389.5631	566.0480
		1	569.1777	685.1907	991.2811	471.6584	568.2058	823.2291	323.2242	389.9532	566.6192
		2	570.3157	686.5662	993.2901	472.6000	569.3444	824.8934	323.8678	390.7322	567.7597
		5	573.7152	690.6757	999.2924	475.4129	572.7461	829.8657	325.7903	393.0593	571.1669
	0.5	0.5	683.2447	821.8676	1187.1371	566.8430	682.3449	987.0391	389.3641	469.3863	680.9720
		1	683.9349	822.7019	1188.3558	567.4145	683.0359	988.0491	389.7552	469.8596	681.6648
		2	685.3132	824.3679	1190.7895	568.5555	684.4157	990.0661	390.5360	470.8046	683.0481
		5	689.4309	829.3454	1198.0604	571.9644	688.5379	996.0918	392.8686	473.6277	687.1809

Table 6.13
Values of frequency parameter Ω for C-F plate vibrating in fundamental mode for ε = 0.3

			η								
			-0.5			0.0			1.0		
			μ			μ			μ		
α	β	p	-0.5	0.0	1.0	-0.5	0.0	1.0	-0.5	0.0	1.0
-0.5	0	0.5	6.0443	6.7885	8.5369	4.8677	5.4702	6.8864	3.1433	3.5361	4.4604
		1	6.3679	7.1862	9.1376	5.1283	5.7908	7.3717	3.3114	3.7434	4.7755
		2	6.9700	7.9215	10.2310	5.6130	6.3836	8.2552	3.6243	4.1268	5.3493
		5	8.5232	9.7992	12.9543	6.8637	7.8976	10.4572	4.4311	5.1061	6.7808
	0.5	0.5	5.8131	6.5269	8.2354	4.6749	5.2510	6.6307	3.0110	3.3846	4.2799
		1	6.3069	7.1384	9.1759	5.0729	5.7443	7.3904	3.2683	3.7040	4.7730
		2	7.1857	8.2122	10.7744	5.7816	6.6111	8.6832	3.7270	4.2659	5.6138
		5	9.2999	10.7455	14.3775	7.4887	8.6591	11.6035	4.8340	5.5968	7.5204
0	-0.5	0.5	7.1484	8.0169	10.0517	5.7653	6.4695	8.1205	3.7330	4.1937	5.2746
		1	7.5284	8.4824	10.7501	6.0715	6.8452	8.6853	3.9310	4.4370	5.6419
		2	8.2356	9.3439	12.0239	6.6414	7.5403	9.7156	4.2992	4.8873	6.3123
		5	10.0608	11.5468	15.2087	8.1119	9.3178	12.2927	5.2490	6.0384	7.9899
	0	0.5	6.7803	7.6019	9.5574	5.4579	6.1218	7.7027	3.5214	3.9528	4.9809
		1	7.3211	8.2692	10.5766	5.8939	6.6604	8.5265	3.8036	4.3019	5.5162
		2	8.2926	9.4542	12.3355	6.6776	7.6173	9.9495	4.3111	4.9226	6.4424
		5	10.6646	12.2978	16.3877	8.5930	9.9163	13.2341	5.5532	6.4169	8.5872
	0.5	0.5	6.7932	7.6315	9.6653	5.4642	6.1407	7.7825	3.5208	3.9592	5.0239
		1	7.5190	8.5315	11.0520	6.0497	6.8671	8.9031	3.8998	4.4300	5.7518
		2	8.7709	10.0574	13.3086	7.0603	8.1000	10.7296	4.5550	5.2306	6.9416
		5	11.6526	13.4873	18.1144	9.3907	10.8772	14.6309	6.0704	7.0405	9.4960
0.5	-0.5	0.5	7.7690	8.7010	10.9106	6.2590	7.0130	8.8012	4.0447	4.5356	5.7007
		1	8.3602	9.4285	12.0150	6.7360	7.6004	9.6943	4.3536	4.9167	6.2817
		2	9.4288	10.7300	13.9409	7.5984	8.6518	11.2531	4.9124	5.5991	7.2971
		5	12.0637	13.8890	18.4459	9.7262	11.2061	14.9053	6.2926	7.2598	9.6826
	0	0.5	7.6997	8.6369	10.8961	6.1973	6.9542	8.7793	3.9978	4.4891	5.6743
		1	8.4693	9.5890	12.3565	6.8183	7.7229	9.9600	4.4001	4.9876	6.4418
		2	9.8131	11.2257	14.7761	7.9033	9.0456	11.9188	5.1037	5.8468	7.7183
		5	12.9612	14.9780	20.0543	10.4488	12.0835	16.2031	6.7587	7.8263	10.5229
	0.5	0.5	7.8148	8.7891	11.1780	6.2869	7.0731	9.0011	4.0520	4.5613	5.8112
		1	8.7765	9.9820	13.0146	7.0628	8.0361	10.4856	4.5546	5.1858	6.7760
		2	10.3979	11.9538	15.9139	8.3725	9.6301	12.8337	5.4046	6.2219	8.3070
		5	14.0186	16.2433	21.8608	11.3033	13.1066	17.6664	7.3136	8.4915	11.4771

Table 6.14
Values of frequency parameter Ω for C-F plate vibrating in second mode for ε = 0.3

			η								
			-0.5			0.0			1.0		
			μ			μ			μ		
α	β	p	-0.5	0.0	1.0	-0.5	0.0	1.0	-0.5	0.0	1.0
-0.5	0	0.5	31.7562	36.8899	49.5800	26.4272	30.7404	41.4256	18.2270	21.2579	28.7993
		1	32.1621	37.3807	50.3031	26.7633	31.1470	42.0253	18.4563	21.5357	29.2102
		2	32.9580	38.3428	51.7188	27.4221	31.9440	43.1994	18.9056	22.0801	30.0146
		5	35.2315	41.0888	55.7478	29.3032	34.2180	46.5413	20.1870	23.6324	32.3039
	0.5	0.5	38.2893	44.2558	58.8992	32.0071	37.0413	49.4195	22.2698	25.8369	34.6409
		1	38.9445	45.0632	60.1385	32.5501	37.7109	50.4485	22.6414	26.2958	35.3481
		2	40.2228	46.6361	62.5425	33.6098	39.0157	52.4457	23.3667	27.1905	36.7220
		5	43.8331	51.0618	69.2392	36.6043	42.6902	58.0172	25.4179	29.7129	40.5636
0	-0.5	0.5	36.8705	42.8698	57.7042	30.6634	35.7021	48.1899	21.1216	24.6596	33.4687
		1	37.3159	43.4068	58.4901	31.0322	36.1468	48.8414	21.3731	24.9634	33.9147
		2	38.1899	44.4601	60.0305	31.7555	37.0191	50.1182	21.8662	25.5588	34.7885
		5	40.6900	47.4718	64.4269	33.8234	39.5119	53.7618	23.2740	27.2588	37.2807
	0	0.5	43.3180	50.1132	66.7983	36.1871	41.9180	56.0176	25.1462	29.2037	39.2256
		1	44.0029	50.9530	68.0742	36.7545	42.6142	57.0765	25.5343	29.6805	39.9526
		2	45.3416	52.5926	70.5571	37.8638	43.9738	59.1379	26.2930	30.6118	41.3689
		5	49.1380	57.2296	77.5248	41.0107	47.8208	64.9290	28.4459	33.2487	45.3541
	0.5	0.5	49.3461	56.9246	75.4684	41.3234	47.7287	63.4292	28.8524	33.4064	44.6085
		1	50.3005	58.1096	77.3164	42.1148	48.7119	64.9642	29.3946	34.0810	45.6647
		2	52.1573	60.4100	80.8826	43.6549	50.6214	67.9290	30.4504	35.3923	47.7074
		5	57.3648	66.8283	90.6983	47.9786	55.9560	76.1058	33.4191	39.0631	53.3607
0.5	-0.5	0.5	48.3769	56.0099	74.7612	40.3890	46.8240	62.6647	28.0333	32.5856	43.8383
		1	49.0950	56.8871	76.0831	40.9839	47.5510	63.7613	28.4399	33.0832	44.5905
		2	50.5005	58.6022	78.6611	42.1481	48.9726	65.9006	29.2357	34.0563	46.0587
		5	54.4974	63.4696	85.9332	45.4594	53.0082	71.9395	31.4986	36.8190	50.2076
	0	0.5	54.4127	62.8162	83.3860	45.5407	52.6408	70.0518	31.7624	36.8069	49.2228
		1	55.3906	64.0252	85.2546	46.3514	53.6435	71.6033	32.3175	37.4944	50.2894
		2	57.2969	66.3776	88.8732	47.9320	55.5953	74.6098	33.4002	38.8335	52.3585
		5	62.6685	72.9784	98.9130	52.3889	61.0775	82.9652	36.4563	42.6003	58.1244
	0.5	0.5	60.1843	69.3442	91.7193	50.4534	58.2031	77.1648	35.3008	40.8215	54.3748
		1	61.4449	70.9166	94.1948	51.4989	59.5080	79.2217	36.0177	41.7175	55.7912
		2	63.8918	73.9608	98.9540	53.5294	62.0359	83.1800	37.4110	43.4551	58.5214
		5	70.7193	82.4023	111.9405	59.2018	69.0571	94.0077	41.3112	48.2938	66.0206

Table 6.15
Values of frequency parameter Ω for C-F plate vibrating in third mode for $\varepsilon = 0.3$

			η								
			-0.5			0.0			1.0		
			μ			μ			μ		
α	β	p	-0.5	0.0	1.0	-0.5	0.0	1.0	-0.5	0.0	1.0
-0.5	0	0.5	85.2318	100.0268	137.2933	71.6576	84.2164	115.9267	50.4675	59.4823	82.3553
		1	85.6673	100.5458	138.0319	72.0221	84.6509	116.5452	50.7219	59.7858	82.7881
		2	86.5305	101.5745	139.4961	72.7443	85.5119	117.7715	51.2256	60.3870	83.6456
		5	89.0604	104.5904	143.7911	74.8597	88.0349	121.3670	52.6991	62.1465	86.1580
	0.5	0.5	111.4454	130.2455	177.2663	94.2673	110.3314	150.6063	67.2077	78.8939	108.3376
		1	112.0779	131.0039	178.3612	94.7960	110.9652	151.5209	67.5763	79.3358	108.9749
		2	113.3316	132.5072	180.5313	95.8439	112.2217	153.3338	68.3066	80.2117	110.2382
		5	117.0074	136.9146	186.8914	98.9156	115.9049	158.6474	70.4460	82.7780	113.9407
0	-0.5	0.5	97.4814	114.5185	157.4947	81.8626	96.3083	132.8377	57.5200	67.8650	94.1545
		1	97.9669	115.0964	158.3151	82.2689	96.7923	133.5252	57.8036	68.2032	94.6359
		2	98.9290	116.2419	159.9417	83.0742	97.7514	134.8882	58.3654	68.8731	95.5900
		5	101.7491	119.6007	164.7141	85.4326	100.5620	138.8855	60.0084	70.8338	98.3851
	0	0.5	124.0546	145.1226	197.8946	104.8023	122.7814	167.9268	74.5291	87.5740	120.4934
		1	124.7385	145.9411	199.0712	105.3744	123.4661	168.9109	74.9283	88.0520	121.1806
		2	126.0941	147.5637	201.4038	106.5082	124.8234	170.8618	75.7193	88.9994	122.5428
		5	130.0691	152.3220	208.2430	109.8318	128.8027	176.5816	78.0362	91.7752	126.5356
	0.5	0.5	148.3658	173.1580	235.0123	125.7832	147.0217	200.1399	90.0867	105.6172	144.6544
		1	149.2423	174.2114	236.5406	126.5156	147.9017	201.4154	90.5972	106.2304	145.5421
		2	150.9797	176.2992	239.5691	127.9673	149.6459	203.9433	91.6088	107.4457	147.3018
		5	156.0735	182.4195	248.4411	132.2234	154.7590	211.3510	94.5735	111.0079	152.4597
0.5	-0.5	0.5	136.5965	159.9285	218.4471	115.2742	135.1636	185.1719	81.7972	96.1956	132.5801
		1	137.3316	160.8072	219.7063	115.8894	135.8991	186.2260	82.2269	96.7095	133.3173
		2	138.7890	162.5494	222.2028	117.1089	137.3572	188.3157	83.0782	97.7281	134.7784
		5	143.0627	167.6589	229.5249	120.6836	141.6323	194.4440	85.5713	100.7124	139.0616
	0	0.5	161.2408	188.3342	256.0150	136.5508	159.7347	217.7931	97.5836	114.4974	157.0681
		1	162.1695	189.4484	257.6256	137.3273	160.6663	219.1388	98.1254	115.1474	158.0066
		2	164.0104	191.6572	260.8178	138.8665	162.5129	221.8062	99.1990	116.4357	159.8668
		5	169.4084	198.1335	270.1729	143.3792	167.9271	229.6247	102.3450	120.2116	165.3197
	0.5	0.5	184.6504	215.3327	291.7699	156.7611	183.0865	248.8317	112.5837	131.8943	180.3657
		1	185.7700	216.6799	293.7301	157.6964	184.2116	250.4669	113.2355	132.6782	181.5030
		2	187.9893	219.3502	297.6144	159.5505	186.4418	253.7075	114.5274	134.2318	183.7574
		5	194.4951	227.1764	308.9893	164.9859	192.9791	263.2015	118.3140	138.7858	190.3649

Table 6.16

Values of frequency parameter Ω for C-F plate vibrating in fundamental mode for ε = 0.5

			η								
			-0.5			0.0			1.0		
			μ			μ			μ		
α	β	p	-0.5	0.0	1.0	-0.5	0.0	1.0	-0.5	0.0	1.0
-0.5	0	0.5	10.6031	12.3537	16.7354	8.4573	9.8565	13.3596	5.3685	6.2601	8.4936
		1	10.8494	12.6628	17.2232	8.6541	10.1035	13.7499	5.4937	6.4174	8.7426
		2	11.3253	13.2581	18.1560	9.0342	10.5793	14.4961	5.7357	6.7205	9.2189
		5	12.6408	14.8932	20.6799	10.0854	11.8866	16.5163	6.4050	7.5539	10.5094
	0.5	0.5	11.5067	13.3786	18.0711	9.1727	10.6671	14.4142	5.8163	6.7666	9.1502
		1	11.9438	13.9321	18.9623	9.5219	11.1095	15.1269	6.0387	7.0483	9.6047
		2	12.7663	14.9671	20.6048	10.1793	11.9370	16.4410	6.4574	7.5758	10.4433
		5	14.9220	17.6478	24.7453	11.9033	14.0819	19.7570	7.5570	8.9450	12.5638
0	-0.5	0.5	12.8708	14.9955	20.3095	10.2726	11.9721	16.2242	6.5287	7.6133	10.3285
		1	13.1438	15.3372	20.8457	10.4907	12.2453	16.6532	6.6676	7.7873	10.6025
		2	13.6731	15.9979	21.8763	10.9136	12.7735	17.4780	6.9368	8.1240	11.1292
		5	15.1476	17.8288	24.6961	12.0920	14.2376	19.7355	7.6873	9.0575	12.5718
	0	0.5	13.5494	15.7532	21.2700	10.8051	12.5653	16.9728	6.8562	7.9764	10.7828
		1	14.0042	16.3273	22.1885	11.1686	13.0243	17.7076	7.0876	8.2689	11.2516
		2	14.8672	17.4110	23.9012	11.8583	13.8908	19.0781	7.5271	8.8214	12.1267
		5	17.1669	20.2696	28.3143	13.6975	16.1781	22.6126	8.7001	10.2815	14.3870
	0.5	0.5	14.6982	17.0784	23.0521	11.7178	13.6179	18.3877	7.4313	8.6393	11.6730
		1	15.3582	17.9162	24.4078	12.2452	14.2875	19.4718	7.7671	9.0659	12.3643
		2	16.5855	19.4624	26.8665	13.2262	15.5239	21.4391	8.3922	9.8542	13.6202
		5	19.7315	23.3714	32.8915	15.7432	18.6528	26.2659	9.9987	11.8529	16.7086
0.5	-0.5	0.5	15.6482	18.1941	24.5616	12.4831	14.5175	19.6071	7.9261	9.2220	12.4656
		1	16.1238	18.7929	25.5145	12.8632	14.9963	20.3695	8.1683	9.5272	12.9522
		2	17.0316	19.9308	27.3062	13.5889	15.9063	21.8035	8.6307	10.1076	13.8681
		5	19.4798	22.9724	31.9975	15.5469	18.3401	25.5611	9.8795	11.6613	16.2714
	0	0.5	16.6745	19.3728	26.1349	13.2966	15.4514	20.8525	8.4364	9.8072	13.2446
		1	17.3482	20.2259	27.5087	13.8350	16.1333	21.9513	8.7793	10.2418	13.9457
		2	18.6113	21.8150	30.0289	14.8448	17.4041	23.9681	9.4229	11.0523	15.2335
		5	21.9008	25.9023	36.3314	17.4765	20.6756	29.0173	11.1024	13.1419	18.4641
	0.5	0.5	17.9168	20.8131	28.0892	14.2845	16.5965	22.4060	9.0598	10.5297	14.2243
		1	18.8032	21.9396	29.9169	14.9927	17.4969	23.8675	9.5109	11.1034	15.1564
		2	20.4387	24.0012	33.1976	16.3002	19.1455	26.4926	10.3442	12.1547	16.8324
		5	24.5733	29.1355	41.0983	19.6088	23.2559	32.8233	12.4567	14.7812	20.8844

Table 6.17
Values of frequency parameter Ω for C-F plate vibrating in second mode for ε = 0.5

			η								
			-0.5			0.0			1.0		
			μ			μ			μ		
α	β	p	-0.5	0.0	1.0	-0.5	0.0	1.0	-0.5	0.0	1.0
-0.5	0	0.5	57.7642	69.0074	98.2796	47.2061	56.4330	80.4819	31.4597	37.6601	53.8561
		1	58.0524	69.3651	98.8334	47.4405	56.7240	80.9326	31.6144	37.8524	54.1542
		2	58.6245	70.0752	99.9318	47.9058	57.3016	81.8265	31.9216	38.2340	54.7454
		5	60.3079	72.1631	103.1559	49.2751	59.0004	84.4512	32.8258	39.3565	56.4819
	0.5	0.5	77.1022	91.8220	129.9560	63.1850	75.2977	106.7093	42.3399	50.5232	71.7874
		1	77.6192	92.4712	130.9857	63.6057	75.8262	107.5479	42.6180	50.8728	72.3428
		2	78.6426	93.7558	133.0200	64.4386	76.8720	109.2049	43.1689	51.5649	73.4408
		5	81.6338	97.5042	138.9329	66.8739	79.9249	114.0241	44.7805	53.5868	76.6375
0	-0.5	0.5	67.0013	80.1142	114.2958	54.7144	65.4684	93.5326	36.4101	43.6271	62.5032
		1	67.3097	80.4959	114.8827	54.9651	65.7787	94.0101	36.5756	43.8320	62.8188
		2	67.9223	81.2538	116.0477	55.4632	66.3952	94.9580	36.9043	44.2390	63.4453
		5	69.7274	83.4861	119.4744	56.9310	68.2108	97.7464	37.8730	45.4380	65.2887
	0	0.5	86.5034	103.1015	146.1505	70.8433	84.4936	119.9339	47.4111	56.6225	80.5877
		1	87.0329	103.7642	147.1940	71.2740	85.0328	120.7835	47.6958	56.9791	81.1501
		2	88.0822	105.0768	149.2585	72.1278	86.1011	122.4646	48.2601	57.6857	82.2633
		5	91.1565	108.9177	155.2807	74.6299	89.2282	127.3707	49.9147	59.7551	85.5147
	0.5	0.5	105.2082	125.1809	176.8458	86.2876	102.7360	145.3261	57.9132	69.0431	97.9187
		1	105.9773	126.1499	178.3925	86.9135	103.5248	146.5860	58.3272	69.5652	98.7536
		2	107.4984	128.0650	181.4443	88.1517	105.0842	149.0724	59.1465	70.5977	100.4020
		5	111.9332	133.6379	190.2842	91.7634	109.6244	156.2796	61.5382	73.6067	105.1863
0.5	-0.5	0.5	95.8725	114.3488	162.3171	78.4709	93.6579	133.1284	52.4562	62.6939	89.3580
		1	96.4173	115.0288	163.3814	78.9140	94.2111	133.9947	52.7490	63.0596	89.9312
		2	97.4977	116.3766	165.4892	79.7930	95.3078	135.7106	53.3296	63.7846	91.0668
		5	100.6690	120.3287	171.6538	82.3733	98.5244	140.7309	55.0350	65.9119	94.3914
	0	0.5	114.7359	136.6050	193.2266	94.0539	112.0558	158.7118	63.0624	75.2326	106.8378
		1	115.5140	137.5828	194.7787	94.6871	112.8516	159.9757	63.4811	75.7591	107.6749
		2	117.0545	139.5171	197.8447	95.9408	114.4263	162.4730	64.3102	76.8012	109.3297
		5	121.5559	145.1604	206.7545	99.6056	119.0224	169.7345	66.7357	79.8452	114.1464
	0.5	0.5	133.1576	158.3546	223.4782	109.2601	130.0201	183.7280	73.3969	87.4567	123.9008
		1	134.1809	159.6462	225.5475	110.0930	131.0717	185.4137	73.9480	88.1529	125.0182
		2	136.2038	162.1974	229.6270	111.7399	133.1493	188.7378	75.0380	89.5289	127.2227
		5	142.0926	169.6086	241.4188	116.5365	139.1882	198.3536	78.2156	93.5327	133.6086

Table 6.18
Values of frequency parameter Ω for C-F plate vibrating in third mode for ε = 0.5

			η								
			-0.5			0.0			1.0		
			μ			μ			μ		
α	β	p	-0.5	0.0	1.0	-0.5	0.0	1.0	-0.5	0.0	1.0
-0.5	0	0.5	155.7863	187.4681	270.9892	128.2589	154.4556	223.5985	86.7765	104.6540	151.9524
		1	156.0734	187.8198	271.5179	128.4936	154.7432	224.0309	86.9333	104.8462	152.2413
		2	156.6458	188.5211	272.5724	128.9618	155.3168	224.8932	87.2461	105.2294	152.8175
		5	158.3503	190.6094	275.7114	130.3557	157.0245	227.4602	88.1772	106.3703	154.5328
	0.5	0.5	222.4003	266.9759	384.0164	183.7618	220.7592	318.0187	125.2293	150.6701	217.7125
		1	222.8635	267.5458	384.8813	184.1402	221.2246	318.7246	125.4815	150.9803	218.1826
		2	223.7871	268.6822	386.6054	184.8945	222.1526	320.1319	125.9844	151.5988	219.1199
		5	226.5357	272.0635	391.7341	187.1397	224.9141	324.3187	127.4813	153.4397	221.9090
0	-0.5	0.5	177.3266	213.5605	309.2022	145.8419	175.7710	254.8650	98.4670	118.8479	172.8383
		1	177.6417	213.9461	309.7804	146.0998	176.0865	255.3381	98.6393	119.0589	173.1547
		2	178.2702	214.7151	310.9333	146.6140	176.7157	256.2815	98.9830	119.4796	173.7858
		5	180.1418	217.0052	314.3665	148.1451	178.5894	259.0907	100.0064	120.7323	175.6647
	0	0.5	245.0761	294.3972	424.0394	202.3045	243.2002	350.8257	137.6001	165.6655	239.7052
		1	245.5679	295.0015	424.9537	202.7064	243.6940	351.5726	137.8683	165.9950	240.2033
		2	246.5487	296.2066	426.7767	203.5080	244.6787	353.0617	138.4032	166.6520	241.1965
		5	249.4681	299.7932	432.2006	205.8938	247.6097	357.4928	139.9954	168.6078	244.1523
	0.5	0.5	309.4643	371.2633	533.3586	255.9543	307.3014	442.1454	174.7744	210.1580	303.3085
		1	310.1313	372.0848	534.6079	256.4989	307.9721	443.1647	175.1374	210.6048	303.9868
		2	311.4611	373.7227	537.0982	257.5848	309.3092	445.1966	175.8610	211.4956	305.3392
		5	315.4181	378.5953	544.5042	260.8162	313.2875	451.2402	178.0148	214.1465	309.3628
0.5	-0.5	0.5	267.4707	321.4903	463.6180	220.6086	265.3622	383.2528	149.8006	180.4606	261.4227
		1	267.9911	322.1291	464.5820	221.0341	265.8844	384.0406	150.0848	180.8094	261.9487
		2	269.0290	323.4028	466.5041	221.8826	266.9257	385.6117	150.6514	181.5048	262.9976
		5	272.1187	327.1945	472.2244	224.4087	270.0255	390.2875	152.3382	183.5749	266.1198
	0	0.5	332.6314	399.2656	574.1909	274.9079	330.2291	475.6325	187.4311	225.4929	325.7787
		1	333.3273	400.1218	575.4899	275.4763	330.9284	476.6929	187.8102	225.9592	326.4852
		2	334.7148	401.8288	578.0793	276.6098	332.3226	478.8070	188.5661	226.8888	327.8938
		5	338.8439	406.9081	585.7818	279.9832	336.4717	485.0962	190.8162	229.6558	332.0854
	0.5	0.5	396.0711	475.0049	681.9246	327.7686	393.3916	565.6279	224.0619	269.3372	388.4625
		1	396.9417	476.0778	683.5581	328.4794	394.2673	566.9603	224.5355	269.9204	389.3489
		2	398.6775	478.2168	686.8142	329.8966	396.0133	569.6164	225.4797	271.0833	391.1161
		5	403.8418	484.5795	696.4959	334.1134	401.2075	577.5155	228.2898	274.5435	396.3732

Table 6.19
Minimum number of collocation points for convergence of frequency parameter Ω

Method	$\eta = 1.0$, $\mu = -0.5$		$\eta = -0.5$, $\mu = 1.0$	
	$\varepsilon = 0.3$	$\varepsilon = 0.5$	$\varepsilon = 0.3$	$\varepsilon = 0.5$
C-C				
NDQM	16	17	19	18
DQM	16	16	17	17
C-S				
NDQM	16	15	18	17
DQM	19	18	21	19
C-F				
NDQM	17	14	19	16
DQM	22	20	25	20

Table 6.20

Comparison of frequency parameter Ω for homogeneous ($\mu = 0.0$, $\eta = 0.0$) uniform thickness ($\alpha = 0.0$, $\beta = 0.0$) annular plates for $\varepsilon = 0.3$, $\nu_\theta = 0.3$

Boundary	Mode	p = 1.0		p = 5.0	
C-C	I	45.3462 45.3371° 45.2‡ 45.37•	45.3462* 45.348† 45.34623♠	48.3540 48.3321°	48.3540* 48.358†
	II	125.3621 125.6191° 125‡	125.3621* 125.404†	129.6030 129.8250°	129.6030* 129.646†
	III	246.1573 246.6994°	246.1563*	250.9706 251.4816°	250.9695*
C-S	I	29.9777 29.9689° 29.9‡	29.9777* 29.979† 30.03679♠	33.2692 33.2528°	33.2692* 33.271†
	II	100.4228 100.6065° 100‡	100.4228* 100.445†	104.7739 104.9319°	104.7739* 104.799†
	III	211.1294 211.5629°	211.1291*	216.0447 216.4574°	216.0444*
C-F	I	6.6604 6.6542° 6.66‡	6.6604* 6.662† 6.70117♠	9.9163 9.9073°	9.9115* 9.917†
	II	42.6142 42.6156° 42.6‡	42.6141* 42.619†	47.8208 47.8100°	47.8284* 47.826†
	III	123.4661 123.5739°	123.4662*	128.8027 128.8986°	128.7846*

* Values taken from Sharma[2006].
° Values taken from Verma[1987].
† Values taken from Gorman[1982].
‡ Values taken from Leissa[1969].
♠ Values taken from Lorrando et al.[1994].
• Values taken from Avalos and laura[1979].

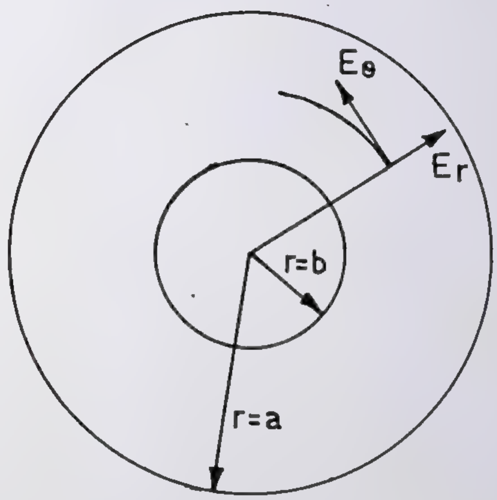

Fig. 6 : Geometry of polar orthotropic annular plate

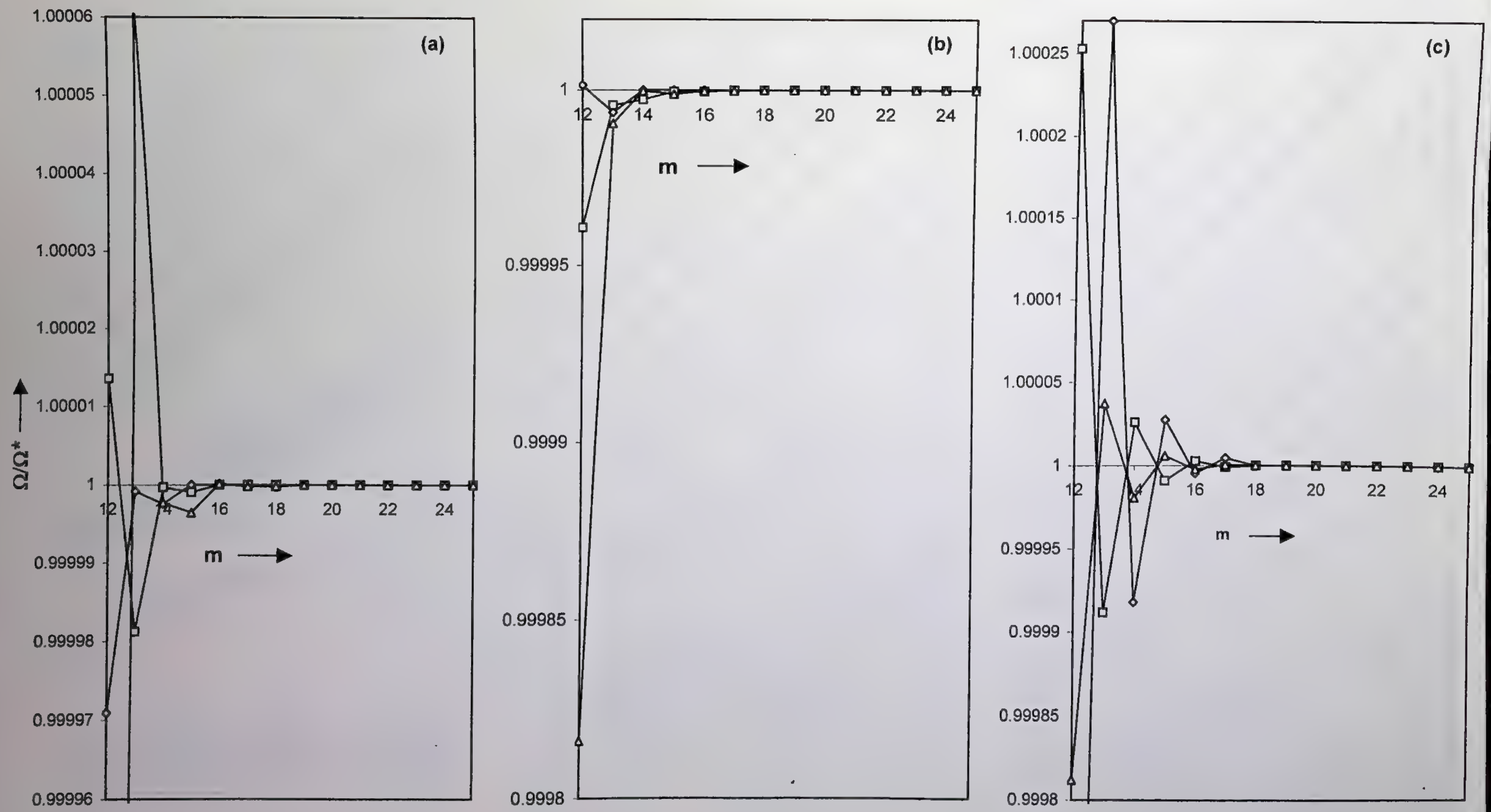

Fig. 6.1 : Convergence of the normalized frequency parameter for Ω/Ω^* for the first three modes of vibration with grid refinement for $\eta = -0.5$, $\mu = 1.0$, $p = 5.0$, $\alpha = 0.5$, $\beta = 0.5$, $\varepsilon = 0.3$ for (a) C-C (b)C-S and (c) C-F plate.
—◊—. fundamental mode; —□—. second mode; —Δ—. third mode. Ω^*- the DQ results using 25 grid points.

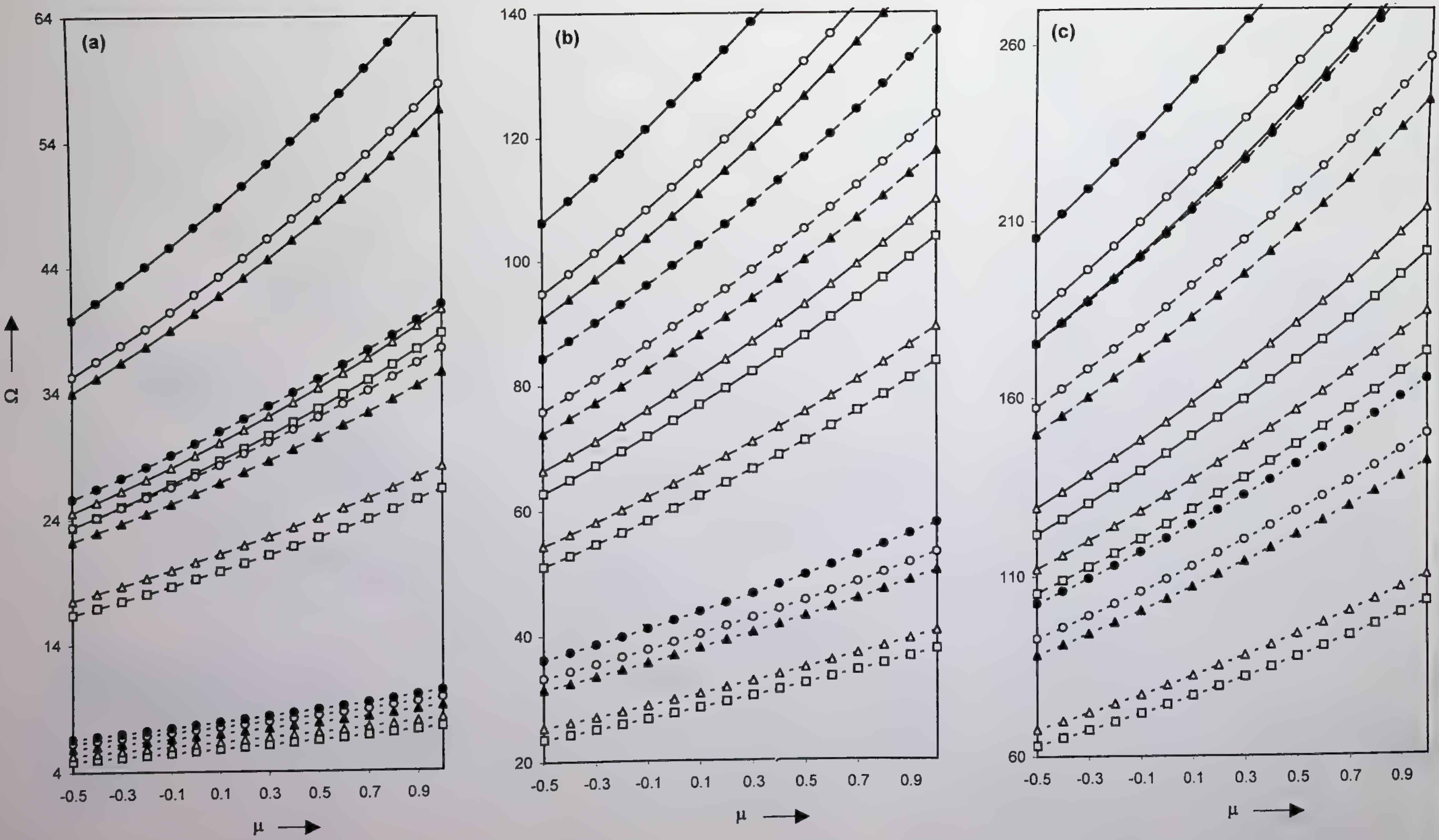

Fig. 6.2 : Frequency parameter for C-C, C-S and C-F plates vibrating in (a) fundamental (b) second and (c) third mode for $\eta = 1.0$, $p = 5.0$, $\varepsilon = 0.3$.
———, C-C ; – – – – – –, C-S ; ------------, C-F.
□, $\alpha = -0.3$, $\beta = 0$; Δ, $\alpha = 0$, $\beta = -0.3$; ▲, $\alpha = 0$, $\beta = 0.3$; ○, $\alpha = 0.3$, $\beta = 0$; ●, $\alpha = 0.3$, $\beta = 0.3$.

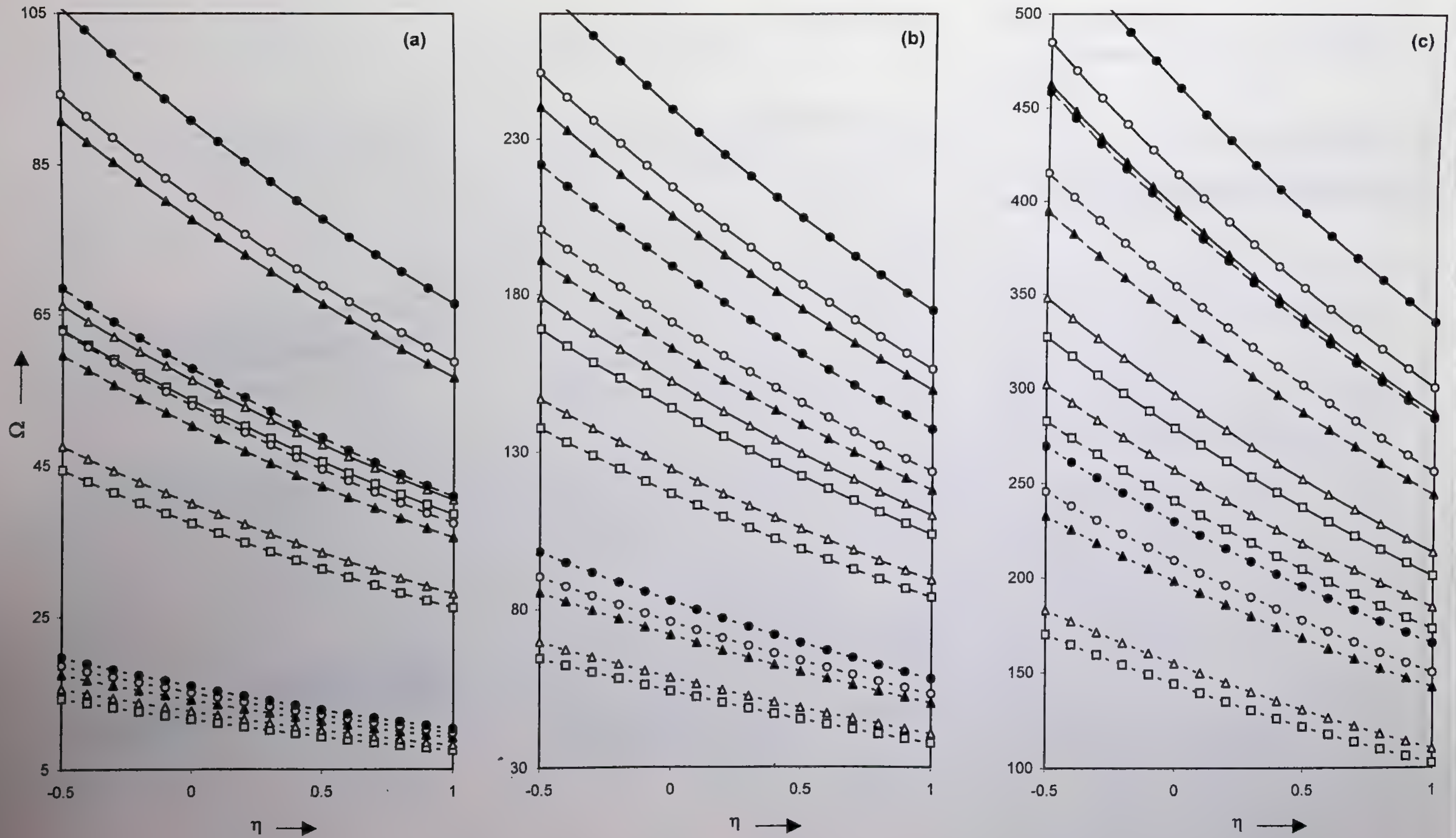

Fig. 6.3 : Frequency parameter for C-C, C-S and C-F plates vibrating in (a) fundamental (b) second and (c) third mode for $\mu = 1.0$, $p = 5.0$, $\varepsilon = 0.3$.
———, C-C ; – – – – – –, C-S ; ------------, C-F.
□, $\alpha = -0.3$, $\beta = 0$; Δ, $\alpha = 0$, $\beta = -0.3$; ▲, $\alpha = 0$, $\beta = 0.3$; ○, $\alpha = 0.3$, $\beta = 0$; ●, $\alpha = 0.3$, $\beta = 0.3$.

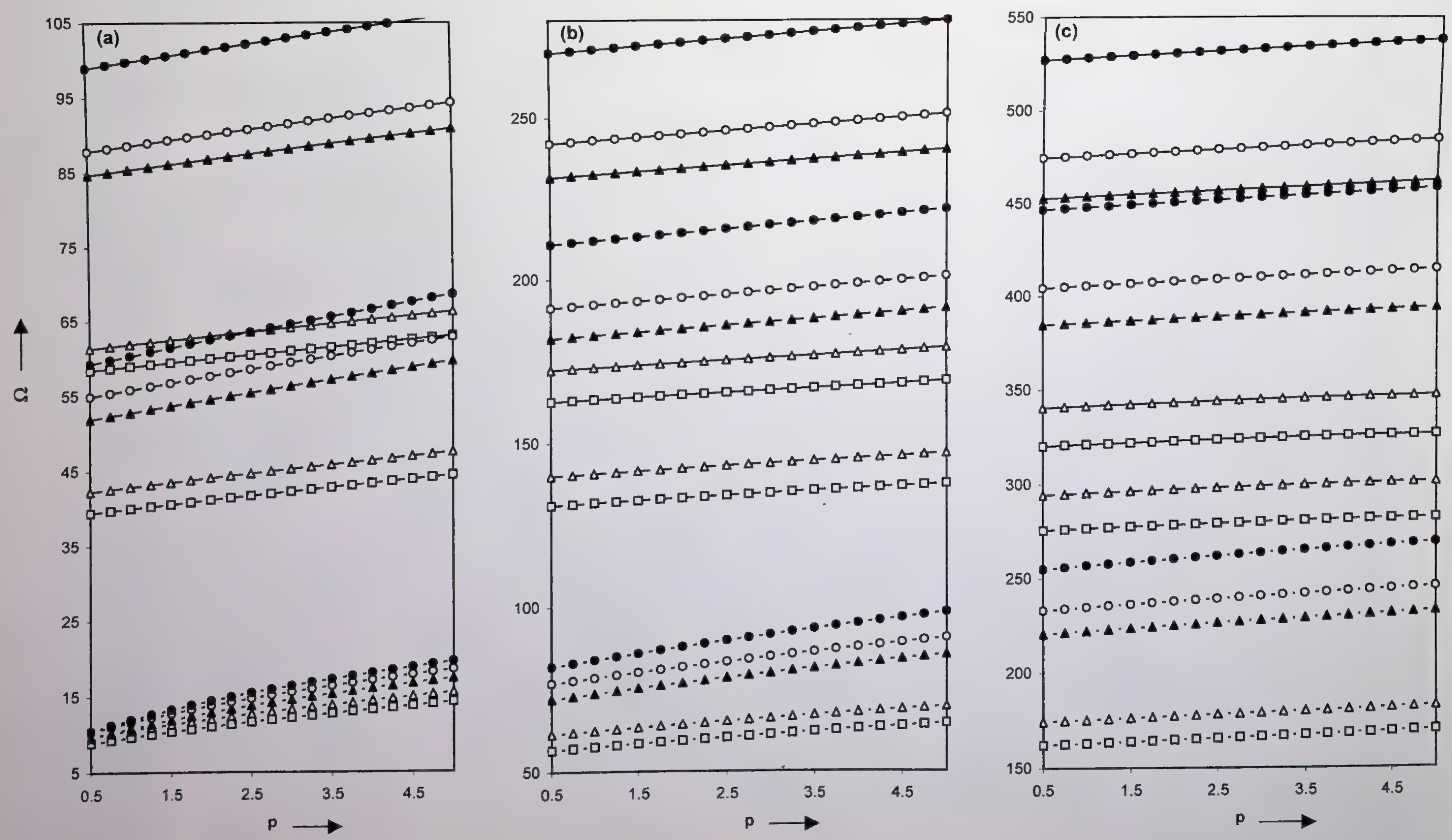

Fig. 6.4 : Frequency parameter for C-C, C-S and C-F plates vibrating in (a) fundamental (b) second and (c) third mode for $\mu = 1.0$, $\eta = -0.5$, $\varepsilon = 0.3$.
———, C-C ; – – – – – –, C-S ; ------------, C-F.
□, $\alpha = -0.3$, $\beta = 0$; Δ, $\alpha = 0$, $\beta = -0.3$; ▲, $\alpha = 0$, $\beta = 0.3$; ○, $\alpha = 0.3$, $\beta = 0$; ●, $\alpha = 0.3$, $\beta = 0.3$.

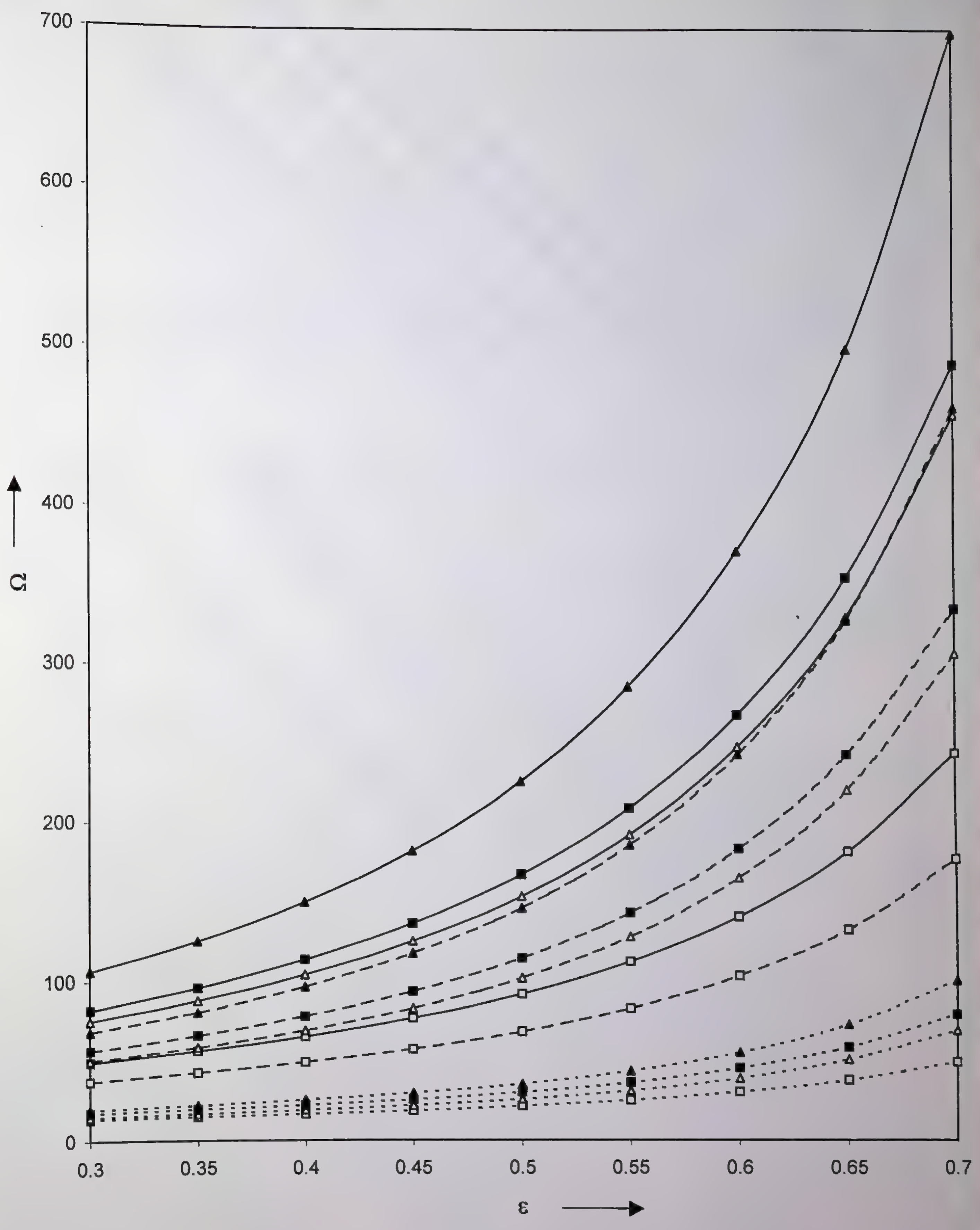

Fig. 6.5: Frequency parameter for C-C, C-S and C-F plates vibrating in fundamental mode for $\mu = 1.0$.
$\eta = -0.5$, $p = 5.0$. ———, C-C ; – – – – – –, C-S ; ------------, C-F.
□, $\alpha = -0.3$, $\beta = -0.3$; Δ, $\alpha = -0.3$, $\beta = 0.3$; ■, $\alpha = 0.3$, $\beta = -0.3$; ▲, $\alpha = 0.3$, $\beta = 0.3$.

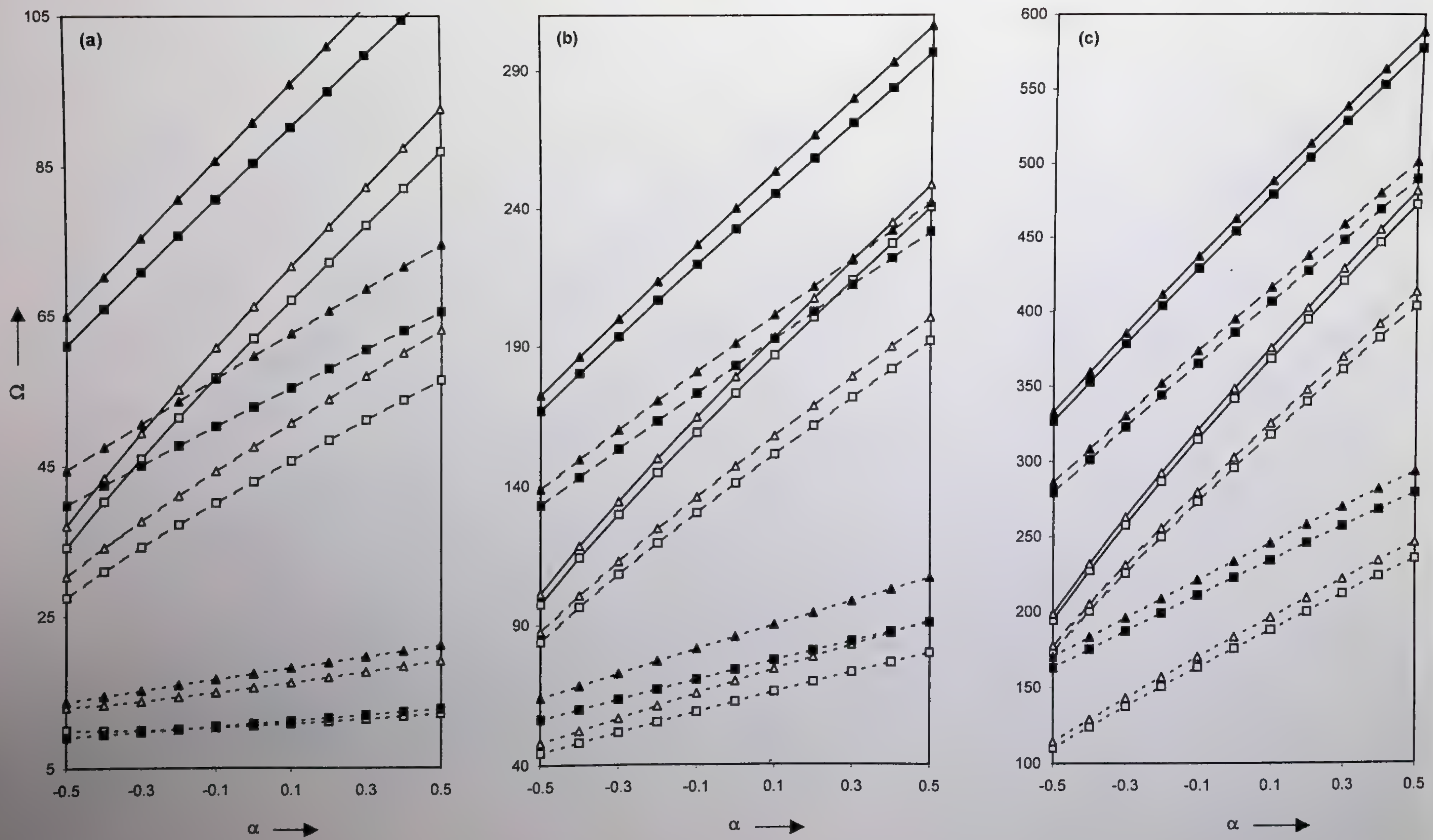

Fig. 6.6 : Frequency parameter for C-C, C-S and C-F plates vibrating in (a) fundamental (b) second and (c) third mode for $\mu = 1.0$, $\eta = -0.5$, $\varepsilon = 0.3$.
———, C-C ; – – – – – –, C-S ; ------------, C-F.
□, $\beta = -0.3$, $p = 1.0$; Δ, $\beta = -0.3$, $p = 5.0$; ■, $\beta = 0.3$, $p = 1.0$; ▲, $\beta = 0.3$, $p = 5.0$.

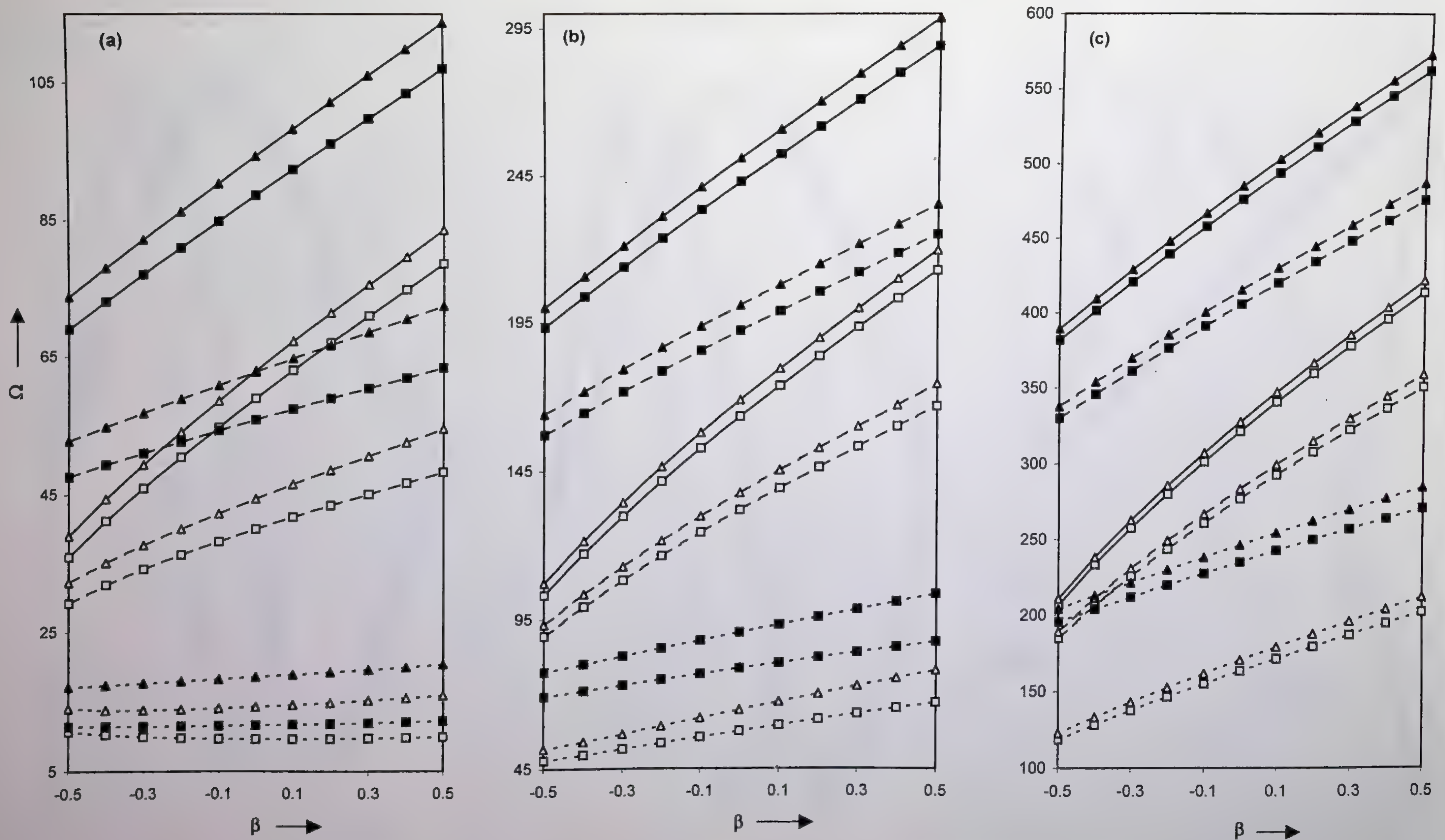

Fig. 6.7 : Frequency parameter for C-C, C-S and C-F plates vibrating in (a) fundamental (b) second and (c) third mode for $\mu = 1.0$, $\eta = -0.5$, $\varepsilon = 0.3$.
———, C-C ; – – – – – –, C-S ; ------------, C-F.
□, $\alpha = -0.3$, $p = 1.0$; Δ, $\alpha = -0.3$, $p = 5.0$: ■, $\alpha = 0.3$, $p = 1.0$; ▲, $\alpha = 0.3$, $p = 5.0$.

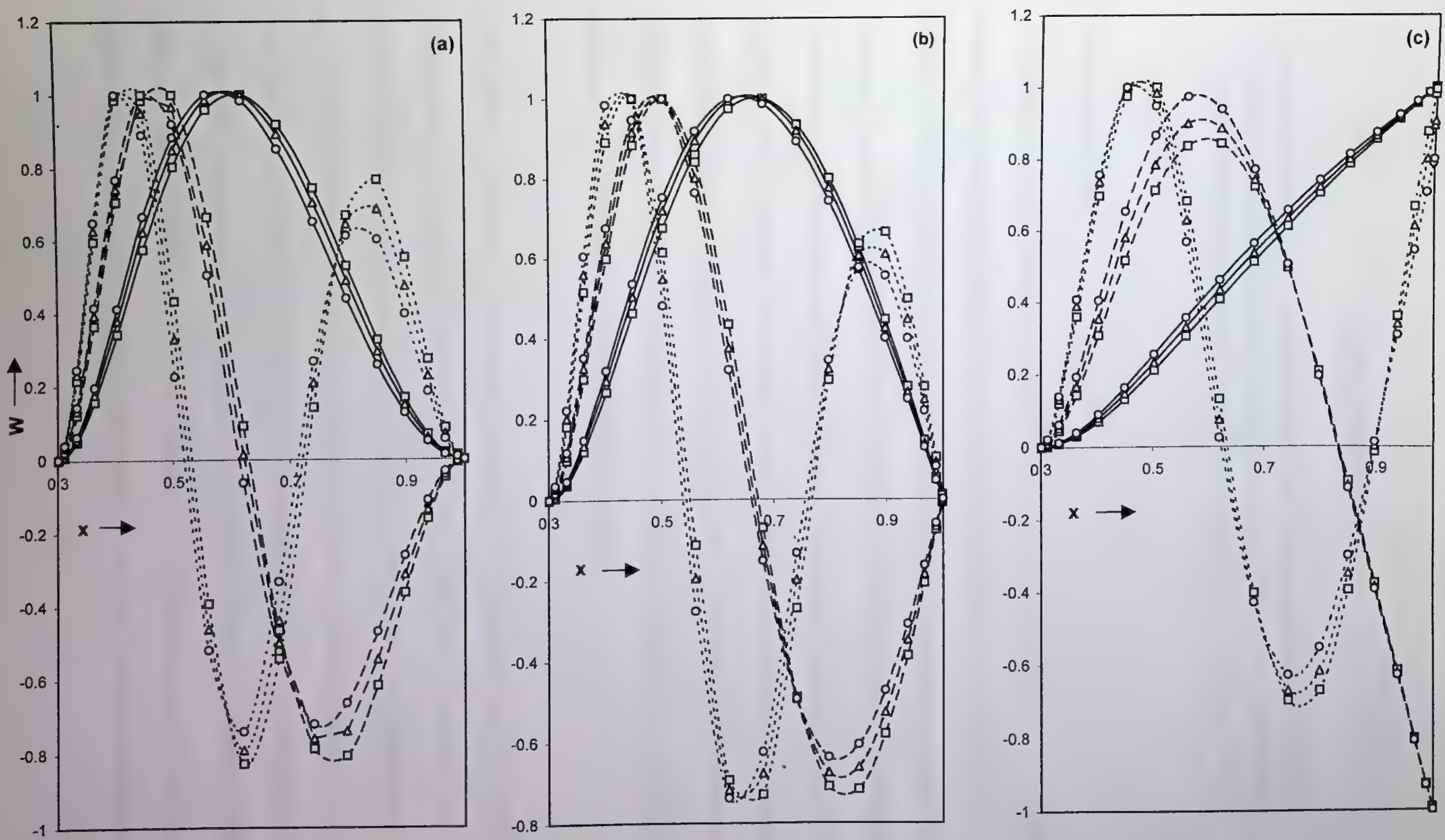

Fig. 6.8 : Normalized displacements for the first three modes of vibration for (a) C-C (b) C-S and (c) C-F plate for $\mu = 1.0$, $\eta = -0.5$, $p = 5.0$, $\varepsilon = 0.3$.
———, fundamental mode; – – – –, second mode; --------, third mode. □, $\alpha = 0$, $\beta = 0$; Δ, $\alpha = 0.3$, $\beta = 0$; ○, $\alpha = 0.3$, $\beta = 0.3$.

CHAPTER VII

AXISYMMETRIC VIBRATIONS OF NON-HOMOGENEOUS POLAR ORTHOTROPIC ANNULAR PLATES OF VARIABLE THICKNESS RESTING ON AN ELASTIC FOUNDATION

1. INTRODUCTION

The desirability of high strength materials for structural components used in mechanical, aerospace and ocean engineering has led to the development of fibre-reinforced materials. The increasing use of high technology composite materials, which are lighter and stronger than the conventional materials, has necessitated the study of vibrational behaviour of polar orthotropic (a special case of anisotropy) plates. Further, the use of plate type structural components under high-temperature environments, particularly in space shuttle and high-speed aircraft, demands that non-homogeneity of the material be taken into account to predict their natural frequencies. Furthermore, the problem of plates resting on an elastic foundation has achieved great importance in modern technology and foundation engineering.

In this chapter, an analysis of axisymmetric vibrations of non-homogeneous polar orthotropic annular plates of exponentially varying thickness and resting on a Winkler type elastic foundation has been presented employing classical plate theory. The non-homogeneity of the plate material is assumed to arise due to variation of Young's moduli and density which are taken to vary exponentially with the radial coordinate. This type of orthotropy and non-homogeneity arises during the fabrication of fibre-reinforced plastic structures, which use fibres with different moduli and strength properties. The differential equation governing the motion of such plates has been solved numerically by using the Chebyshev polynomials for

three different combinations of boundary conditions at the two edges. The effect of various plate parameters, namely radii ratio, thickness variation and orthotropy together with elastic foundation has been analysed on the vibrational behaviour of the plate for the first three modes of vibration. Mode shapes for a specified plate have been presented. The accuracy of the method employed has been verified by comparing the results with those available in literature for isotropic and polar orthotropic plates of uniform thickness.

2. BASIC PLATE EQUATION

Consider an annular plate of thickness $h(r)$, inner and outer radii b and a, respectively. referred to cylindrical polar coordinates (r, θ, z) with its axis as the line $r = 0$ and middle surface as the plane $z = 0$. Let the plate rest on a Winkler type elastic foundation with modulus of foundation K_f.

Energy Variations

The work done by the foundation is given by

$$W_{foundation} = \frac{1}{2}\int_b^a \int_0^{2\pi} K_f w^2 r \, d\theta \, dr . \tag{7.2.1}$$

Taking the variation of $W_{foundation}$, we get

$$\delta W_{foundation} = \int_b^a \int_0^{2\pi} K_f w \delta w r \, d\theta \, dr . \tag{7.2.2}$$

Equation of Motion

To obtain the governing equation of motion by Hamilton's energy principle, the contribution of work done by the foundation given by equation (7.2.2) is incorporated together with the strain energy (6.2.11) and kinetic energy (6.2.12) in

$$\delta L = \delta T - \delta W - \delta W_{foundation}$$

where

$$\delta W = \int_b^a \int_0^{2\pi} \left\{ \begin{array}{l} D_r \left[\dfrac{\partial^2 w}{\partial r^2} \dfrac{\partial^2 (\delta w)}{\partial r^2} + \dfrac{1}{2} \dfrac{\upsilon_\theta}{r} \left(\dfrac{\partial w}{\partial r} \dfrac{\partial^2 (\delta w)}{\partial r^2} + \dfrac{\partial (\delta w)}{\partial r} \dfrac{\partial^2 w}{\partial r^2} \right) \right] \\ + D_\theta \left[\dfrac{1}{r^2} \dfrac{\partial w}{\partial r} \dfrac{\partial (\delta w)}{\partial r} + \dfrac{1}{2} \dfrac{\upsilon_r}{r} \left(\dfrac{\partial w}{\partial r} \dfrac{\partial^2 (\delta w)}{\partial r^2} + \dfrac{\partial (\delta w)}{\partial r} \dfrac{\partial^2 w}{\partial r^2} \right) \right] \end{array} \right\} r \, d\theta \, dr, \tag{7.2.3}$$

$$\delta T = \int_b^a \int_0^{2\pi} \rho h \left(\frac{\partial w}{\partial t} \frac{\partial (\delta w)}{\partial t} \right) r \, d\theta \, dr . \tag{7.2.4}$$

Now considering $\delta W + \delta W_{foundation} - \delta T$, Hamilton's principle

$$\delta \int_{t_1}^{t_2} L \, dt = 0 \tag{7.2.5}$$

gives

$$\int_b^a \int_0^{2\pi} \int_{t_1}^{t_2} \left[\begin{array}{l} D_r \left\{ \dfrac{\partial^2 w}{\partial r^2} \dfrac{\partial^2 (\delta w)}{\partial r^2} + \dfrac{1}{r} \upsilon_\theta \left(\dfrac{\partial^2 w}{\partial r^2} \dfrac{\partial (\delta w)}{\partial r} + \dfrac{\partial w}{\partial r} \dfrac{\partial^2 (\delta w)}{\partial r^2} \right) + \dfrac{\upsilon_\theta}{\upsilon_r} \dfrac{1}{r^2} \dfrac{\partial w}{\partial r} \dfrac{\partial (\delta w)}{\partial r} \right\} \\ + K_f w \delta w - \rho h \dfrac{\partial w}{\partial t} \dfrac{\partial (\delta w)}{\partial t} \end{array} \right] r \, dt \, d\theta \, dr = 0, \tag{7.2.6}$$

where $D_r = \dfrac{E_r h^3}{12(1 - \upsilon_r \upsilon_\theta)}$, $D_\theta = \dfrac{E_\theta h^3}{12(1 - \upsilon_r \upsilon_\theta)}$ and $D_r \upsilon_\theta = \upsilon_r D_\theta$.

Integrating equation (7.2.6) by parts, the integrated part gives the boundary conditions while the remaining triple integrals are

$$\int_b^a\int_0^{2\pi}\int_{t_1}^{t_2}\left[\frac{1}{12(1-\upsilon_r\upsilon_\theta)}\left\{\begin{array}{l}\frac{\partial^2}{\partial r^2}\left(E_r h^3 r\frac{\partial^2 w}{\partial r^2}\right)-\upsilon_\theta\frac{\partial}{\partial r}\left(E_r h^3\frac{\partial^2 w}{\partial r^2}\right)\\ +\upsilon_\theta\frac{\partial^2}{\partial r^2}\left(E_r h^3\frac{\partial w}{\partial r}\right)-\frac{\upsilon_\theta}{\upsilon_r}\frac{\partial}{\partial r}\left(E_r\frac{h^3}{r}\frac{\partial w}{\partial r}\right)\end{array}\right\}+K_f wr+\rho hr\frac{\partial^2 w}{\partial t^2}\right]\delta w\, dt\, d\theta\, dr=0 \cdot \tag{7.2.7}$$

Expression (7.2.7) will be satisfied only when the coefficient of δw is zero and hence

$$\frac{1}{12(1-\upsilon_r\upsilon_\theta)}\left\{\begin{array}{l}\frac{\partial^2}{\partial r^2}\left(E_r h^3 r\frac{\partial^2 w}{\partial r^2}\right)-\upsilon_\theta\frac{\partial}{\partial r}\left(E_r h^3\frac{\partial^2 w}{\partial r^2}\right)\\ +\upsilon_\theta\frac{\partial^2}{\partial r^2}\left(E_r h^3\frac{\partial w}{\partial r}\right)-\frac{\upsilon_\theta}{\upsilon_r}\frac{\partial}{\partial r}\left(E_r\frac{h^3}{r}\frac{\partial w}{\partial r}\right)\end{array}\right\}+K_f wr+\rho hr\frac{\partial^2 w}{\partial t^2}=0 \quad , \tag{7.2.8}$$

which is the required plate equation of motion.

For a non-homogeneous plate, the simplification of the above equation (7.2.8) leads to

$$\begin{aligned}&E_r\frac{\partial^4 w}{\partial r^4}+\frac{2}{r}\left[E_r+r\frac{dE_r}{dr}\right]\frac{\partial^3 w}{\partial r^3}\\ &\quad+\frac{1}{r^2}\left[-E_\theta+r(2+\upsilon_\theta)\frac{dE_r}{dr}+r^2\frac{d^2E_r}{dr^2}\right]\frac{\partial^2 w}{\partial r^2}\\ &\quad+\frac{1}{r^3}\left[E_\theta-r\frac{dE_\theta}{dr}+r^2\upsilon_\theta\frac{d^2E_r}{dr^2}\right]\frac{\partial w}{\partial r}\\ &\quad+\frac{12(1-\upsilon_r\upsilon_\theta)}{h^3}K_f w+\frac{12(1-\upsilon_r\upsilon_\theta)\rho}{h^2}\frac{\partial^2 w}{\partial t^2}=0.\end{aligned} \tag{7.2.9}$$

Introducing non-dimensional variables $x=\frac{r}{a}, \overline{w}=\frac{w}{a}, \overline{h}=\frac{h}{a}$, together with exponential variation in thickness along radial direction i.e. $\overline{h}=h_o e^{\alpha x}$ and exponential variation in rigidities and density for non-homogeneity of the plate material as follows :

$$E_r = E_1 e^{\mu x}, \qquad E_\theta = E_2 e^{\mu x}, \qquad \rho=\rho_o e^{\eta x}, \tag{7.2.10}$$

equation (7.2.9) now reduces to

$$P_0\frac{d^4W}{dx^4}+P_1\frac{d^3W}{dx^3}+P_2\frac{d^2W}{dx^2}+P_3\frac{dW}{dx}+P_4W=0, \tag{7.2.11}$$

where $\overline{w}(x,t)=W(x)e^{i\omega t}$ (for harmonic vibrations), ω is the radian frequency, h_0, ρ_0 are the thickness and density of the plate at $x = 0$, μ is the non-homogeneity parameter, η is the density parameter and α is the taper parameter,

$$P_0=1, \qquad P_1=\frac{2}{x}\{1+(\mu+3\alpha)x\},$$

$$P_2=\frac{1}{x^2}\left\{-p+(2+\upsilon_\theta)(\mu+3\alpha)x+(\mu+3\alpha)^2x^2\right\},$$

$$P_3=\frac{1}{x^3}\left\{p-p(\mu+3\alpha)x+\upsilon_\theta(\mu+3\alpha)^2x^2\right\},$$

$$P_4=-\Omega^2e^{(\eta-\mu-2\alpha)x}+\frac{12K(1-\upsilon_r\upsilon_\theta)}{h_0^3}e^{-(\mu+3\alpha)x},$$

$$p=\frac{E_2}{E_1}, \qquad K=\frac{aK_f}{E_1}, \qquad \Omega^2=\frac{12\rho_0a^2\omega^2(1-\upsilon_r\upsilon_\theta)}{E_1h_0^2}$$

and E_1, E_2 are moduli in radial and tangential directions at $x = 0$, respectively.

Equation (7.2.11) together with the boundary conditions at the edges $x = \varepsilon$ and $x = 1$, where $\varepsilon = b/a$, constitutes a two point boundary value problem in the range (ε, 1), which has been solved by Chebyshev collocation technique.

3. METHOD OF SOLUTION : CHEBYSHEV COLLOCATION TECHNIQUE

By taking a new independent variable

$$y \equiv \frac{1}{(1-\varepsilon)}\{2x-(1+\varepsilon)\}\ , \tag{7.3.1}$$

the range $\varepsilon \le x \le 1$ is transformed to $-1 \le y \le 1$, which is the applicability range of the technique. Equation (7.2.11) now reduces to

$$A_0\frac{d^4W}{dy^4}+A_1\frac{d^3W}{dy^3}+A_2\frac{d^2W}{dy^2}+A_3\frac{dW}{dy}+A_4W=0\ , \tag{7.3.2}$$

where $A_i = \xi^{4-i}\,P_i$, i = 0, 1, 2, 3, 4 and $\xi = 2/(1\text{-}\varepsilon)$. According to Chebyshev collocation method, we assume

$$\frac{d^4W}{dy^4}=\sum_{k=0}^{m-5}c_{k+5}\,T_k\ , \tag{7.3.3}$$

and its successive integrations lead to

$$W=c_1+c_2T_1+c_3T_1^1+c_4T_1^2+\sum_{k=0}^{m-5}c_{k+5}\,T_k^4\ , \tag{7.3.4}$$

where c_j (j = 1, 2,..., m) are unknown constants, T_k (k = 0, 1, 2,..., m-5) are Chebyshev polynomials and T_k^j represents the j^{th} integral of T_k.

Substitution of W and its derivatives in equation (7.3.2) gives

$$\begin{aligned} A_4c_1+(A_4T_1)c_2+(A_4T_1^1+A_3T_1+A_2)c_3+(A_4T_1^2+A_3T_1^1+A_2T_1+A_1)c_4 \\ +(A_4T_1^3+A_3T_1^2+A_2T_1^1+A_1T_1+A_0)c_5 \\ +\sum_{i=1}^{m-5}(A_4T_i^4+A_3T_i^3+A_2T_i^2+A_1T_i^1+A_0T_i)c_{i+5}=0. \end{aligned} \tag{7.3.5}$$

Satisfaction of this resultant equation at (m-4) collocation points given by

$$y_k = \cos\left(\frac{2k-1}{m-4}\frac{\pi}{2}\right), \quad k = 1, 2, \ldots, m\text{-}4 , \tag{7.3.6}$$

provides a set of (m-4) equations in terms of the unknowns c_j (j = 1, 2,..., m), which can be written in matrix form as

$$[B][C^*] = 0 \quad , \tag{7.3.7}$$

where B and C^* are matrices of order (m-4) x m and m x 1, respectively.

4. BOUNDARY CONDITIONS AND FREQUENCY EQUATIONS

By satisfying the relations $W = dW/dy = 0$, $W = \xi\,(d^2W/dy^2) + (\upsilon_\theta\,/x)(dW/dy) = 0$ and $\xi(d^2W/dy^2) + (\upsilon_\theta\,/x)(dW/dy) = \xi^2(d^3W/dy^3) + (\xi/x)(d^2W/dy^2) - (p/x^2)(dW/dy) = 0$ for clamped, simply supported and free edge respectively, a set of four homogeneous equations are obtained for (i) C-C (ii) C-S and (iii) C-F. These equations together with the field equations (7.3.7) give a complete set of m equations in m unknowns, whose non-trivial solution for C-C, C-S and C-F plates respectively leads to

$$\begin{vmatrix} B \\ B^{CC} \end{vmatrix} = 0 , \quad \begin{vmatrix} B \\ B^{CS} \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} B \\ B^{CF} \end{vmatrix} = 0 , \tag{7.4.1, 7.4.2, 7.4.3}$$

where B^{CC}, B^{CS} and B^{CF} are matrices of order 4 x m.

5. NUMERICAL RESULTS AND DISCUSSIONS

The frequency equations (7.4.1-7.4.3) provide the values of the frequency parameter Ω for various values of plate parameters. First three natural frequencies of vibration have been computed for non-homogeneity parameter μ = -0.5(0.1)1.0, density parameter η = -0.5(0.1)1.0, radii ratio ε = 0.3(0.05)0.5, rigidity parameter p = 0.5, 1.0, 2.0, 3.0, 4.0, 5.0, taper constant α =

-0.5(0.1)0.5 and foundation parameter K = 0.0(0.01)0.1 for all the three boundary conditions for υ_θ = 0.3. The numerical values show a consistent improvement with the increase of the number of collocation points. In all the computations, the number of collocation points has been taken as m = 19, since further increase in m does not improve the results except at the fourth place of decimal (Figures 7.1(a,b,c)). The value of the thickness h_0 at the origin has been taken as 0.1.

The numerical results are presented in Figures (7.2-7.8) and Tables (7.1-7.12). Tables (7.1-7.12) present the values of first three frequency parameters for μ = -0.5, 0.0, 1.0, η = -0.5, 0.0. 1.0, p = 0.5, 1.0, 2.0, α = -0.5, 0.0, 0.5, ε = 0.3, 0.5 and K = 0.0, 0.02 for C-C, C-S and C-F plates. From the results, it is found that the frequency parameter for C-S plate is higher than that for C-F plate but it is less than that for C-C plate irrespective of value of other plate parameters. The frequency parameter is found to increase with increasing values of non-homogeneity parameter μ, taper parameter α, rigidity parameter p, foundation parameter K as well as radii ratio ε, while it decreases with increasing values of density parameter η.

Figure 7.2a shows the plots for frequency parameter Ω versus non-homogeneity parameter μ for fixed radii ratio ε = 0.3, density parameter η = -0.5, rigidity parameter p = 2.0, for two values of taper parameter α = -0.5, 0.5 and foundation parameter K = 0.0, 0.02 for all the three plates vibrating in the fundamental mode. It is observed that frequency parameter increases with increasing value of non-homogeneity parameter μ. The effect of elastic foundation increases the frequencies for all the three plates. The effect decreases with increasing values of non-homogeneity parameter μ for all the three cases. It can also be seen that the effect is more pronounced in case of C-F plate in comparison to those of C-S and C-C plates. From Figure

7.2b, showing the plots for Ω versus μ in the second mode of vibration, it is observed that the foundation parameter increases the frequencies with increasing value of μ except that the rate of increase of Ω for all the plates is higher than that in the fundamental mode. A similar behaviour can be seen from Figure 7.2c, when the plate is vibrating in third mode of vibration. Figures 7.2(a,b,c) depict that the effect of foundation parameter decreases with the increase in the number of modes for all the three plates whatever are the values of plate parameters. Figures 7.3(a,b,c) show the effect of density parameter η on the frequency parameter Ω for μ =1.0, ε = 0.3, p = 2.0, α = -0.5, 0.5 and K = 0.0, 0.02 for C-C, C-S and C-F plates vibrating in fundamental, second and third mode respectively. The frequency parameter is found to decrease with increasing value of η. This rate of decrease increases in the order of boundary conditions C-F, C-S, C-C for same set of other plate parameters and also with increasing number of modes. The effect of foundation is more pronounced for negative value of α as compared to that for its positive value. The effect of foundation decreases with increasing value of η for all the three cases.

Figures 7.4(a,b,c) depict the effect of taper parameter α on first three frequency parameters Ω for ε = 0.3, μ = 1.0, η = -0.5, p = 0.5, 5.0 and K = 0.0, 0.02 for all the three plates. The frequency parameter Ω is found to increase with increasing value of the taper parameter α except in case of C-F plates for p = 0.5 and K = 0.02. In this case, the frequency parameter is found to decrease with increasing value of α. The rate of increase of Ω for $\alpha > 0$ is higher as compared to that for $\alpha < 0$ for all the three boundary conditions. This rate of increase reduces in the order of boundary conditions C-C, C-S, C-F for same set of other plate parameters. The effect of foundation decreases with increasing value of α.

Figures 7.5(a,b,c) show the behaviour of frequency parameter Ω with rigidity parameter p for $\mu = 1.0$, $\eta = -0.5$, $\varepsilon = 0.3$, $\alpha = -0.5$, 0.5 and two values of $K = 0.0$, 0.02 for C-C, C-S and C-F plates for first three modes of vibration. The frequencies are found to increase as the plate becomes more and more stiff in the tangential direction ($p > 1$) as compared to radial direction ($p < 1$). Thus, the increase in orthotropy increases the frequencies keeping all other plate parameters fixed. Figures 7.6(a,b,c) depict the effect of radii ratio ε on first three frequency parameter Ω for all the three plates for fixed values of $\mu = 1.0$, $\eta = -0.5$, $p = 2.0$, $\alpha = -0.5$, 0.5 and $K = 0.0$, 0.02. It is seen that by increasing the hole size of the plate, frequency increases. This effect is more pronounced in the case of C-C plate as compared to that of C-S and C-F plates. The effect of foundation decreases with increasing value of ε and becomes almost negligible for $\varepsilon > 0.35$ in the case of C-C plate, for $\varepsilon > 0.5$ in the case of C-S plate and for $\varepsilon >$ 0.65 in the case of C-F plate.

Figures 7.7(a,b,c) show the effect of foundation parameter K on frequency parameter Ω for $p = 2.0$, $\alpha = 0.5$, $\varepsilon = 0.3$, $\mu = -0.5$, 1.0 and $\eta = -0.5$, 1.0 for plates vibrating in first three modes of vibration. The foundation parameter K increases the frequencies for all the three plates. The rate of increase of Ω with K reduces with the increase in number of modes. The effect of non-homogeneity decreases with increasing value of K, while the effect of density increases with increasing value of K.

Figures 7.8(a,b,c) show the plots for normalised transverse displacements for $\varepsilon = 0.3$, $\mu = 1.0$, $\eta = -0.5$, $p = 2.0$, $\alpha = \pm 0.5$ and $K = 0.02$ for the first three modes of vibration for C-C, C-S and C-F plates, respectively. The radii of the nodal circles decrease as the outer edge becomes

thicker and thicker for all the three boundary conditions. Table 7.13 shows a comparison of results for homogeneous ($\mu = 0.0$, $\eta = 0.0$) isotropic ($p = 1$) and orthotropic ($p = 5$) plates of uniform thickness ($\alpha = 0$) for $\varepsilon = 0.3$ and $K = 0.0, 0.01$ with those of Verma[1987], obtained by quintic spline technique.

Table 7.1

Values of frequency parameter for C-C annular plate for K = 0.0 and ε = 0.3

			η								
			-0.5			0.0			1.0		
			μ			μ			μ		
α	p	Mode	-0.5	0.0	1.0	-0.5	0.0	1.0	-0.5	0.0	1.0
-0.5	0.5	I	32.3461	38.0678	52.6955	27.3816	32.2805	44.8383	19.5205	23.0916	32.2954
		II	89.8514	105.8415	146.5349	76.0917	89.7676	124.6547	54.3266	64.2823	89.8015
		III	176.8262	208.3620	288.4554	149.7781	176.7379	245.3620	107.0093	126.6262	176.7825
	1.0	I	32.6519	38.4225	53.1727	27.6375	32.5781	45.2406	19.6986	23.2997	32.5792
		II	90.2536	106.3125	147.1791	76.4316	90.1662	125.2016	54.5681	64.5663	90.1936
		III	177.2752	208.8876	289.1738	150.1594	177.1848	245.9744	107.2830	126.9478	177.2256
	2.0	I	33.2523	39.1196	54.1113	28.1400	33.1628	46.0317	20.0480	23.7082	33.1370
		II	91.0511	107.2465	148.4572	77.1053	90.9565	126.2864	55.0463	65.1291	90.9710
		III	178.1686	209.9337	290.6039	150.9179	178.0742	247.1933	107.8273	127.5877	178.1073
0	0.5	I	44.8383	52.8299	73.3020	38.0865	44.9520	62.5870	27.3375	32.3762	45.3895
		II	124.6547	146.7286	202.8366	105.8822	124.8192	173.0684	76.0488	89.9194	125.4297
		III	245.3620	288.6718	398.3930	208.4143	245.5462	339.8285	149.7414	176.9155	246.2222
	1.0	I	45.2406	53.2976	73.9350	38.4248	45.3462	63.1230	27.5750	32.6543	45.7712
		II	125.2016	147.3683	203.7096	106.3456	125.3621	173.8117	76.3799	90.3086	125.9659
		III	245.9744	289.3876	399.3684	208.9356	246.1563	340.6619	150.1175	177.3568	246.8282
	2.0	I	46.0317	54.2178	75.1817	39.0899	46.1218	64.1787	28.0417	33.2011	46.5226
		II	126.2864	148.6375	205.4424	107.2648	126.4393	175.2869	77.0361	91.0802	127.0296
		III	247.1933	290.8127	401.3108	209.9731	247.3707	342.3214	150.8656	178.2350	248.0345
0.5	0.5	I	62.5870	73.8338	102.7175	53.3455	63.0408	88.0081	38.5528	45.7177	64.2711
		II	173.0684	203.5608	280.9713	147.4470	173.6860	240.4551	106.5395	125.8768	175.3175
		III	339.8285	399.1892	549.1918	289.4662	340.5077	469.7783	209.1454	246.7144	342.2941
	1.0	I	63.1230	74.4594	103.5711	53.7984	63.5706	88.7344	38.8737	46.0949	64.7932
		II	173.8117	204.4293	282.1533	148.0788	174.4254	241.4647	106.9937	126.4100	176.0504
		III	340.6619	400.1619	550.5126	290.1774	341.3387	470.9097	209.6610	247.3186	343.1209
	2.0	I	64.1787	75.6920	105.2542	54.6901	64.6143	90.1665	39.5053	46.8376	65.8221
		II	175.2869	206.1532	284.5004	149.3326	175.8930	243.4693	107.8945	127.4680	177.5051
		III	342.3214	402.0989	553.1437	291.5933	342.9935	473.1632	210.6874	248.5213	344.7673

Table 7.2
Values of frequency parameter for C-C annular plate for K = 0.02 and ε = 0.3

			η								
			-0.5			0.0			1.0		
			μ			μ			μ		
α	p	Mode	-0.5	0.0	1.0	-0.5	0.0	1.0	-0.5	0.0	1.0
-0.5	0.5	I	37.8156	42.7824	56.1504	32.0185	36.2840	47.7819	22.8262	25.9555	34.4156
		II	91.9773	107.6413	147.8245	77.8824	91.2859	125.7460	55.6051	65.3696	90.5877
		III	177.9185	209.2846	289.1151	150.6960	177.5143	245.9187	107.6651	127.1824	177.1836
	1.0	I	38.6284	43.5827	56.9630	32.7041	36.9597	48.4698	23.3098	26.4334	34.9046
		II	92.5998	108.2994	148.6034	78.4078	91.8424	126.4069	55.9790	65.7666	91.0618
		III	178.4839	209.9087	289.9040	151.1751	178.0440	246.5905	108.0087	127.5634	177.6696
	2.0	I	39.4099	44.4382	58.0207	33.3592	37.6783	49.3619	23.7663	26.9363	35.5344
		II	93.4910	109.3132	149.9389	79.1605	92.6999	127.5403	56.5135	66.3774	91.8742
		III	179.4305	210.9997	291.3664	151.9784	178.9712	247.8367	108.5849	128.2304	178.5709
0	0.5	I	47.7819	55.3417	75.1201	40.5880	47.0902	64.1399	29.1298	33.9136	46.5139
		II	125.7460	147.6540	203.5031	106.8075	125.6050	173.6360	76.7180	90.4895	125.8440
		III	245.9187	289.1438	398.7332	208.8860	245.9466	340.1179	150.0839	177.2071	246.4341
	1.0	I	48.4698	56.0553	75.9333	41.1687	47.6936	64.8299	29.5406	34.3417	47.0068
		II	126.4069	148.3905	204.4459	107.3676	126.2302	174.4388	77.1190	90.9383	126.4235
		III	246.5905	289.9100	399.7450	209.4576	246.5995	340.9823	150.4965	177.6796	247.0628
	2.0	I	49.3619	57.0627	77.2442	41.9192	48.5430	65.9401	30.0677	34.9409	47.7972
		II	127.5403	149.7011	206.2085	108.3279	127.3424	175.9395	77.8050	91.7353	127.5058
		III	247.8367	291.3582	401.7041	210.5181	247.8336	342.6560	151.2614	178.5721	248.2795
0.5	0.5	I	64.1399	75.1547	103.6711	54.6686	64.1681	88.8247	39.5058	46.5325	64.8655
		II	173.6360	204.0437	281.3213	147.9314	174.0987	240.7551	106.8944	126.1801	175.5393
		III	340.1179	399.4356	549.3709	289.7134	340.7185	469.9320	209.3278	246.8704	342.4085
	1.0	I	64.8299	75.9118	104.6202	55.2525	64.8101	89.6328	39.9209	46.9906	65.4470
		II	174.4388	204.9628	282.5400	148.6141	174.8814	241.7962	107.3858	126.7451	176.2955
		III	340.9823	400.4347	550.7109	290.4510	341.5720	471.0798	209.8630	247.4912	343.2475
	2.0	I	65.9401	77.1912	106.3375	56.1905	65.8935	91.0940	40.5853	47.7617	66.4969
		II	175.9395	206.7084	284.9029	149.8896	176.3675	243.8143	108.3025	127.8166	177.7602
		III	342.6560	402.3838	553.3508	291.8790	343.2372	473.3409	210.8983	248.7016	344.8995

Table 7.3

Values of frequency parameter for C-C annular plate for K = 0.0 and ε = 0.5

α	p	Mode	η = -0.5, μ = -0.5	η = -0.5, μ = 0.0	η = -0.5, μ = 1.0	η = 0.0, μ = -0.5	η = 0.0, μ = 0.0	η = 0.0, μ = 1.0	η = 1.0, μ = -0.5	η = 1.0, μ = 0.0	η = 1.0, μ = 1.0
-0.5	0.5	I	61.0806	73.6805	107.1775	50.5270	61.0039	88.8951	34.4836	41.7074	60.9914
		II	168.8308	203.7590	296.4430	139.6976	168.7282	245.8553	95.4255	115.4327	168.7152
		III	331.3912	400.0293	582.0197	274.2416	331.2798	482.6829	187.4063	226.7089	331.2676
	1.0	I	61.2668	73.9034	107.4971	50.6803	61.1876	89.1591	34.5870	41.8316	61.1709
		II	169.0784	204.0572	296.8747	139.9023	168.9749	246.2130	95.5651	115.6011	168.9603
		III	331.6643	400.3580	582.4953	274.4678	331.5524	483.0778	187.5613	226.8959	331.5393
	2.0	I	61.6371	74.3467	108.1329	50.9850	61.5529	89.6844	34.7928	42.0788	61.5281
		II	169.5724	204.6521	297.7361	140.3107	169.4670	246.9268	95.8434	115.9371	169.4491
		III	332.2095	401.0145	583.4454	274.9197	332.0967	483.8665	187.8708	227.2693	332.0818
0	0.5	I	88.8951	107.2912	156.2404	73.6658	88.9892	129.8186	50.4529	61.0555	89.3850
		II	245.8553	296.6008	431.1730	203.7435	245.9860	358.1435	139.6019	168.8048	246.5283
		III	482.6829	582.1937	845.7118	400.0145	482.8273	702.3736	274.1394	331.3668	483.4231
	1.0	I	89.1591	107.6079	156.6965	73.8837	89.2508	130.1962	50.6006	61.2334	89.6431
		II	246.2130	297.0313	431.7954	204.0397	246.3428	358.6601	139.8044	169.0492	246.8832
		III	483.0778	582.6687	846.3980	400.3421	483.2216	702.9440	274.3645	331.6381	483.8164
	2.0	I	89.6844	108.2380	157.6041	74.3171	89.7713	130.9479	50.8946	61.5873	90.1567
		II	246.9268	297.8902	433.0373	204.6307	247.0545	359.6908	140.2084	169.5366	247.5915
		III	483.8665	583.6175	847.7688	400.9965	484.0093	704.0834	274.8140	332.1798	484.6021
0.5	0.5	I	129.8186	156.7721	228.5640	107.7691	130.2603	190.2499	74.0711	89.6889	131.4620
		II	358.1435	431.8937	627.3487	297.2548	358.7421	521.8922	204.3010	246.9401	360.3505
		III	702.3736	846.5019	1227.6890	582.9121	703.0297	1021.0736	400.6280	483.8747	704.7898
	1.0	I	130.1962	157.2261	229.2210	108.0815	130.6363	190.7955	74.2840	89.9458	131.8366
		II	358.6601	432.5148	628.2453	297.6832	359.2576	522.6377	204.5949	247.2944	360.8644
		III	702.9440	847.1874	1228.6776	583.3859	703.5995	1021.8963	400.9543	484.2676	705.3586
	2.0	I	130.9479	158.1299	230.5288	108.7032	131.3848	191.8813	74.7078	90.4572	132.5823
		II	359.6908	433.7542	630.0345	298.5380	360.2863	524.1251	205.1812	248.0012	361.8898
		III	704.0834	848.5567	1230.6523	584.3323	704.7377	1023.5397	401.6061	485.0524	706.4946

Table 7.4

Values of frequency parameter for C-C annular plate for K = 0.02 and ε = 0.5

α	p	Mode	η								
			-0.5			0.0			1.0		
			μ			μ			μ		
			-0.5	0.0	1.0	-0.5	0.0	1.0	-0.5	0.0	1.0
-0.5	0.5	I	64.4320	76.4722	109.1020	53.3013	63.3168	90.4923	36.3769	43.2886	62.0872
		II	170.0759	204.7888	297.1474	140.7251	169.5786	246.4379	96.1274	116.0145	169.1151
		III	332.0281	400.5556	582.3794	274.7665	331.7139	482.9800	187.7650	227.0059	331.4716
	1.0	I	64.9650	76.9864	109.6246	53.7415	63.7418	90.9248	36.6762	43.5779	62.3823
		II	170.4577	205.1980	297.6552	141.0405	169.9170	246.8586	96.3425	116.2457	169.4033
		III	332.3704	400.9415	582.8942	275.0498	332.0336	483.4073	187.9590	227.2252	331.7654
	2.0	I	65.4913	77.5609	110.3519	54.1753	64.2158	91.5260	36.9698	43.8992	62.7915
		II	171.0154	205.8457	298.5527	141.5015	170.4528	247.6022	96.6568	116.6115	169.9126
		III	332.9493	401.6258	583.8634	275.5294	332.6009	484.2117	188.2875	227.6144	332.3187
0	0.5	I	90.4923	108.6159	157.1499	74.9897	90.0881	130.5743	51.3588	61.8089	89.9050
		II	246.4379	297.0832	431.5040	204.2259	246.3857	358.4182	139.9335	169.0801	246.7180
		III	482.9800	582.4397	845.8807	400.2604	483.0310	702.5137	274.3089	331.5074	483.5200
	1.0	I	90.9248	109.0727	157.7025	75.3472	90.4660	131.0323	51.6021	62.0664	90.2183
		II	246.8586	297.5658	432.1621	204.5742	246.7857	358.9644	140.1719	169.3542	247.0935
		III	483.4073	582.9415	846.5853	400.6148	483.4475	703.0993	274.5524	331.7940	483.9239
	2.0	I	91.5260	109.7660	158.6536	75.8434	91.0389	131.8201	51.9390	62.4562	90.7567
		II	247.6022	298.4495	433.4210	205.1899	247.5179	360.0093	140.5930	169.8557	247.8115
		III	484.2117	583.9033	847.9650	401.2822	484.2460	704.2461	275.0109	332.3431	484.7147
0.5	0.5	I	130.5743	157.3985	228.9941	108.3963	130.7807	190.6079	74.5015	90.0466	131.7089
		II	358.4182	432.1214	627.5055	297.4830	358.9314	522.0228	204.4589	247.0714	360.4413
		III	702.5137	846.6181	1227.7692	583.0285	703.1264	1021.1403	400.7089	483.9419	704.8364
	1.0	I	131.0323	157.9191	229.6969	108.7754	131.2120	191.1915	74.7601	90.3415	132.1098
		II	358.9644	432.7672	628.4191	297.9360	359.4675	522.7824	204.7699	247.4399	360.9651
		III	703.0993	847.3163	1228.7665	583.5150	703.7067	1021.9703	401.0440	484.3422	705.4102
	2.0	I	131.8201	158.8529	231.0254	109.4271	131.9854	192.2945	75.2045	90.8701	132.8673
		II	360.0093	434.0183	630.2164	298.8025	360.5059	524.2765	205.3643	248.1534	361.9951
		III	704.2461	848.6917	1230.7454	584.4676	704.8500	1023.6172	401.7001	485.1305	706.5488

Table 7.5
Values of frequency parameter for C-S annular plate for K = 0.0 and ε = 0.3

α	p	Mode	η								
			-0.5			0.0			1.0		
			μ			μ			μ		
			-0.5	0.0	1.0	-0.5	0.0	1.0	-0.5	0.0	1.0
-0.5	0.5	I	22.8811	26.6362	35.9913	19.1529	22.3279	30.2542	13.3457	15.6012	21.2555
		II	73.3806	86.1669	118.4813	61.9295	72.8286	100.4407	43.9243	51.8083	71.8800
		III	153.0759	180.1092	248.5350	129.4428	152.5164	211.0530	92.1903	108.9288	151.5895
	1.0	I	23.2000	27.0125	36.5179	19.4181	22.6414	30.6950	13.5277	15.8174	21.5622
		II	73.7874	86.6464	119.1468	62.2718	73.2326	101.0028	44.1655	52.0936	72.2789
		III	153.5300	180.6430	249.2716	129.8273	152.9689	211.6787	92.4648	109.2525	152.0392
	2.0	I	23.8232	27.7480	37.5476	19.9361	23.2542	31.5566	13.8831	16.2397	22.1614
		II	74.5929	87.5957	120.4652	62.9495	74.0322	102.1160	44.6426	52.6580	73.0687
		III	154.4329	181.7046	250.7371	130.5917	153.8686	212.9233	93.0103	109.8961	152.9334
0	0.5	I	30.2542	35.1598	47.3170	25.3828	29.5386	39.8578	17.7662	20.7298	28.1177
		II	100.4407	117.7873	161.5124	85.0281	99.8632	137.3503	60.6804	71.4833	98.9181
		III	211.0530	247.8676	340.7516	178.9860	210.5047	290.2096	128.2155	151.2224	209.6714
	1.0	I	30.6950	35.6845	48.0689	25.7509	29.9777	40.4898	18.0210	21.0352	28.5611
		II	101.0028	118.4502	162.4348	85.5020	100.4228	138.1307	61.0156	71.8801	99.4741
		III	211.6787	248.6028	341.7656	179.5168	211.1291	291.0723	128.5959	151.6708	210.2936
	2.0	I	31.5566	36.7102	49.5375	26.4703	30.8359	41.7242	18.5190	21.6319	29.4269
		II	102.1160	119.7635	164.2626	86.4404	101.5312	139.6769	61.6791	72.6658	100.5754
		III	212.9233	250.0656	343.7834	180.5724	212.3710	292.7888	129.3523	152.5626	211.5312
0.5	0.5	I	39.8578	46.2108	61.8689	33.5092	38.9002	52.2115	23.5489	27.4063	36.9654
		II	137.3503	160.8394	219.8719	116.6415	136.7984	187.5838	83.7716	98.5517	135.9808
		III	290.2096	340.1811	465.8317	246.8385	289.7545	397.9150	177.8644	209.3870	289.2038
	1.0	I	40.4898	46.9741	62.9993	34.0392	39.5416	53.1655	23.9190	27.8561	37.6398
		II	138.1307	161.7610	221.1582	117.3008	137.5778	188.6737	84.2399	99.1065	136.7600
		III	291.0723	341.1945	467.2288	247.5715	290.6165	399.1053	178.3918	210.0084	290.0650
	2.0	I	41.7242	48.4637	65.2006	35.0742	40.7934	55.0231	24.6415	28.7336	38.9532
		II	139.6769	163.5873	223.7076	118.6067	139.1221	190.8338	85.1670	100.2055	138.3039
		III	292.7888	343.2111	470.0095	249.0300	292.3315	401.4742	179.4407	211.2444	291.7786

Table 7.6
Values of frequency parameter for C-S annular plate for K = 0.02 and ε = 0.3

α	p	Mode	η = -0.5, μ = -0.5	η = -0.5, μ = 0.0	η = -0.5, μ = 1.0	η = 0.0, μ = -0.5	η = 0.0, μ = 0.0	η = 0.0, μ = 1.0	η = 1.0, μ = -0.5	η = 1.0, μ = 0.0	η = 1.0, μ = 1.0
-0.5	0.5	I	30.4134	33.2983	41.1144	25.4675	27.9200	34.5657	17.7458	19.5087	24.2847
		II	76.0096	88.4034	120.0983	64.1326	74.7055	101.8017	45.4868	53.1433	72.8540
		III	154.3466	181.1845	249.3067	130.5076	153.4186	211.7022	92.9485	109.5731	152.0558
	1.0	I	31.3663	34.2476	42.0954	26.2640	28.7144	35.3888	18.2971	20.0601	24.8594
		II	76.6844	89.1117	120.9303	64.6994	75.3015	102.5039	45.8870	53.5652	73.3530
		III	154.9354	181.8323	250.1254	131.0049	153.9669	212.3969	93.3034	109.9653	152.5551
	2.0	I	32.1792	35.1478	43.2454	26.9401	29.4647	36.3514	18.7607	20.5769	25.5286
		II	77.5992	90.1544	122.3163	65.4686	76.1794	103.6740	46.4289	54.1852	74.1834
		III	155.8989	182.9453	251.6278	131.8201	154.9097	213.6726	93.8852	110.6396	153.4716
0	0.5	I	34.5657	38.9222	50.1613	29.0015	32.7006	42.2545	20.2953	22.9458	29.8064
		II	101.8017	118.9466	162.3547	86.1776	100.8438	138.0649	61.5079	72.1914	99.4374
		III	211.7022	248.4191	341.1507	179.5350	210.9717	290.5485	128.6134	151.5618	209.9192
	1.0	I	35.3888	39.7836	51.1696	29.6902	33.4225	43.1025	20.7737	23.4490	30.4017
		II	102.5039	119.7290	163.3640	86.7697	101.5044	138.9190	61.9282	72.6611	100.0470
		III	212.3969	249.2130	342.2071	180.1241	211.6457	291.4472	129.0361	152.0463	210.5677
	2.0	I	36.3514	40.8944	52.6965	30.4939	34.3519	44.3859	21.3296	24.0948	31.3016
		II	103.6740	121.0907	165.2269	87.7561	102.6537	140.4950	62.6262	73.4763	101.1698
		III	213.6726	250.7021	344.2441	181.2060	212.9100	293.1799	129.8115	152.9544	211.8172
0.5	0.5	I	42.2545	48.2933	63.4394	35.5236	40.6527	53.5365	24.9610	28.6380	37.9013
		II	138.0649	161.4500	220.3190	117.2498	137.3190	187.9661	84.2161	98.9333	136.2629
		III	290.5485	340.4702	466.0429	247.1276	290.0016	398.0960	178.0776	209.5698	289.3384
	1.0	I	43.1025	49.2439	64.7094	36.2349	41.4517	54.6082	25.4580	29.1984	38.6589
		II	138.9190	162.4347	221.6514	117.9718	138.1521	189.0955	84.7302	99.5276	137.0713
		III	291.4472	341.5144	467.4625	247.8914	290.8898	399.3055	178.6277	210.2106	290.2140
	2.0	I	44.3859	50.7733	66.9352	37.3109	42.7368	56.4865	26.2088	30.0990	39.9866
		II	140.4950	164.2864	224.2193	119.3030	139.7180	191.2713	85.6757	100.6423	138.6267
		III	293.1799	343.5448	470.2533	249.3637	292.6167	401.6831	179.6868	211.4553	291.9340

Table 7.7

Values of frequency parameter for C-S annular plate for K = 0.0 and ε = 0.5

			η								
			-0.5			0.0			1.0		
			μ			μ			μ		
α	p	Mode	-0.5	0.0	1.0	-0.5	0.0	1.0	-0.5	0.0	1.0
-0.5	0.5	I	43.0059	51.4716	73.6181	35.2859	42.2627	60.5342	23.6864	28.4105	40.8091
		II	137.6838	165.7855	240.0190	113.6435	136.9427	198.5647	77.2562	93.2373	135.6074
		III	286.6693	345.6656	501.7479	236.9466	285.9151	415.6144	161.5485	195.2151	284.5888
	1.0	I	43.2157	51.7268	73.9977	35.4577	42.4720	60.8462	23.8012	28.5507	41.0191
		II	137.9435	166.1002	240.4813	113.8573	137.2020	198.9461	77.4008	93.4129	135.8663
		III	286.9522	346.0077	502.2477	237.1804	286.1979	416.0279	161.7079	195.4080	284.8714
	2.0	I	43.6321	52.2334	74.7511	35.7987	42.8874	61.4654	24.0291	28.8289	41.4357
		II	138.4612	166.7277	241.4032	114.2836	137.7190	199.7065	77.6892	93.7631	136.3825
		III	287.5171	346.6907	503.2456	237.6472	286.7626	416.8536	162.0260	195.7933	285.4358
0	0.5	I	60.5342	72.3876	103.3284	49.7268	59.5060	85.0592	33.4584	40.0940	57.4690
		II	198.5647	238.9272	345.4159	164.1527	197.6720	286.2149	111.9461	135.0130	196.0955
		III	415.6144	500.6737	725.3563	344.0341	414.7427	601.7307	235.2550	284.0166	413.2612
	1.0	I	60.8462	72.7697	103.9049	49.9829	59.8200	85.5340	33.6303	40.3052	57.7899
		II	198.9461	239.3897	346.0964	164.4670	198.0535	286.7766	112.1592	135.2719	196.4776
		III	416.0279	501.1736	726.0865	344.3761	415.1563	602.3353	235.4885	284.2993	413.6751
	2.0	I	61.4654	73.5277	105.0483	50.4912	60.4429	86.4757	33.9715	40.7243	58.4263
		II	199.7065	240.3120	347.4533	165.0937	198.8140	287.8968	112.5839	135.7879	197.2393
		III	416.8536	502.1718	727.5445	345.0589	415.9821	603.5426	235.9547	284.8636	414.5016
0.5	0.5	I	85.0592	101.5994	144.6662	69.9506	83.6097	119.2106	47.1686	56.4549	80.7064
		II	286.2149	344.1404	496.7599	236.9922	285.1771	412.2893	162.1412	195.4120	283.3997
		III	601.7307	724.1783	1047.0980	498.8387	600.7834	869.9408	342.1331	412.6519	599.2658
	1.0	I	85.5340	102.1854	145.5661	70.3412	84.0924	119.9532	47.4319	56.7809	81.2102
		II	286.7766	344.8223	497.7649	237.4556	285.7399	413.1195	162.4559	195.7946	283.9652
		III	602.3353	724.9091	1048.1654	499.3391	601.3885	870.8252	342.4753	413.0660	599.8721
	2.0	I	86.4757	103.3474	147.3490	71.1158	85.0492	121.4246	47.9540	57.4273	82.2084
		II	287.8968	346.1818	499.7686	238.3797	286.8620	414.7748	163.0833	196.5574	285.0926
		III	603.5426	726.3684	1050.2969	500.3382	602.5967	872.5913	343.1585	413.8930	601.0827

Table 7.8

Values of frequency parameter for C-S annular plate for K = 0.02 and ε = 0.5

			η								
			-0.5			0.0			1.0		
			μ			μ			μ		
α	p	Mode	-0.5	0.0	1.0	-0.5	0.0	1.0	-0.5	0.0	1.0
-0.5	0.5	I	47.7936	55.5230	76.4895	39.2165	45.5912	62.8966	26.3249	30.6480	42.4017
		II	139.2244	167.0633	240.8980	114.9107	137.9945	199.2893	78.1176	93.9534	136.1023
		III	287.4093	346.2778	502.1675	237.5555	286.4192	415.9603	161.9636	195.5592	284.8256
	1.0	I	48.4772	56.1848	77.1621	39.7773	46.1345	63.4496	26.7008	31.0127	42.7741
		II	139.6490	167.5151	241.4548	115.2601	138.3666	199.7486	78.3545	94.2058	136.4143
		III	287.7725	346.6864	502.7128	237.8553	286.7567	416.4114	162.1680	195.7895	285.1340
	2.0	I	49.0926	56.8613	78.0367	40.2817	46.6894	64.1685	27.0382	31.3846	43.2580
		II	140.2440	168.2067	242.4209	115.7499	138.9364	200.5454	78.6860	94.5919	136.9554
		III	288.3763	347.4016	503.7328	238.3541	287.3478	417.2553	162.5079	196.1928	285.7108
0	0.5	I	62.8966	74.3717	104.7241	51.6677	61.1373	86.2082	34.7634	41.1924	58.2449
		II	199.2893	239.5288	345.8311	164.7510	198.1692	286.5585	112.3560	135.3541	196.3320
		III	415.9603	500.9605	725.5537	344.3200	414.9799	601.8942	235.4517	284.1801	413.3742
	1.0	I	63.4496	74.9571	105.4442	52.1219	61.6184	86.8013	35.0685	41.5162	58.6456
		II	199.7486	240.0560	346.5562	165.1296	198.6041	287.1572	112.6131	135.6496	196.7395
		III	416.4114	501.4915	726.3053	344.6930	415.4192	602.5166	235.7065	284.4804	413.8004
	2.0	I	64.1685	75.7987	106.6457	52.7120	62.3100	87.7909	35.4647	41.9815	59.3143
		II	200.5454	241.0085	347.9340	165.7864	199.3896	288.2946	113.0584	136.1828	197.5131
		III	417.2553	502.5048	727.7737	345.3909	416.2575	603.7324	236.1831	285.0534	414.6328
0.5	0.5	I	86.2082	102.5633	145.3448	70.8954	84.4029	119.7697	47.8049	56.9897	81.0844
		II	286.5585	344.4262	496.9580	237.2770	285.4143	412.4539	162.3378	195.5760	283.5139
		III	601.8942	724.3141	1047.1919	498.9745	600.8963	870.0190	342.2273	412.7304	599.3203
	1.0	I	86.8013	103.2485	146.3144	71.3832	84.9671	120.5697	48.1336	57.3708	81.6271
		II	287.1572	345.1388	497.9842	237.7711	286.0025	413.3017	162.6736	195.9762	284.0916
		III	602.5166	725.0597	1048.2696	499.4896	601.5136	870.9119	342.5798	413.1530	599.9325
	2.0	I	87.7909	104.4504	148.1247	72.1972	85.9568	122.0637	48.6823	58.0393	82.6406
		II	288.2946	346.5127	499.9978	238.7094	287.1365	414.9653	163.3109	196.7472	285.2247
		III	603.7324	726.5262	1050.4060	500.4959	602.7278	872.6821	343.2680	413.9841	601.1460

Table 7.9
Values of frequency parameter for C-F annular plate for K = 0.0 and ε = 0.3

			η								
			-0.5			0.0			1.0		
			μ			μ			μ		
α	p	Mode	-0.5	0.0	1.0	-0.5	0.0	1.0	-0.5	0.0	1.0
-0.5	0.5	I	5.9584	6.6888	8.4103	4.7964	5.3872	6.7804	3.0948	3.4793	4.3870
		II	33.3063	38.6357	51.7816	27.7528	32.2360	43.3180	54.3784	64.0191	30.1872
		III	91.3767	107.1125	146.6703	76.9521	90.3337	124.0547	107.0435	126.3471	88.4322
	1.0	I	6.3168	7.1300	9.0797	5.0851	5.7429	7.3211	3.2812	3.7095	4.7380
		II	33.7629	39.1905	52.6072	28.1310	32.6957	44.0028	54.6583	64.3534	30.6568
		III	91.8560	107.6845	147.4872	77.3532	90.8125	124.7385	107.3448	126.7047	88.9103
	2.0	I	6.9779	7.9376	10.2817	5.6177	6.3943	8.2926	3.6253	4.1311	5.3692
		II	34.6575	40.2766	54.2206	28.8717	33.5957	45.3415	55.2127	65.0158	31.5748
		III	92.8061	108.8184	149.1068	78.1480	91.7613	126.0942	107.9440	127.4160	89.8578
0	0.5	I	6.7804	7.6019	9.5575	5.4579	6.1218	7.7028	3.5214	3.9528	4.9809
		II	43.3180	50.1131	66.7981	36.1870	41.9179	56.0175	25.1462	29.2036	39.2255
		III	124.0547	145.1228	197.8953	104.8024	122.7815	167.9274	74.5291	87.5740	120.4936
	1.0	I	7.3211	8.2693	10.5767	5.8939	6.6604	8.5266	3.8036	4.3019	5.5163
		II	44.0028	50.9529	68.0739	36.7545	42.6142	57.0763	25.5343	29.6804	39.9525
		III	124.7385	145.9413	199.0721	105.3744	123.4662	168.9116	74.9283	88.0521	121.1809
	2.0	I	8.2926	9.4543	12.3358	6.6777	7.6174	9.9497	4.3111	4.9227	6.4426
		II	45.3415	52.5925	70.5566	37.8638	43.9737	59.1376	26.2930	30.6117	41.3687
		III	126.0942	147.5640	201.4050	106.5082	124.8235	170.8626	75.7193	88.9995	122.5431
0.5	0.5	I	7.7028	8.6468	10.9345	6.1985	6.9607	8.8081	3.9971	4.4914	5.6904
		II	56.0175	64.6267	85.6848	46.9095	54.1867	72.0184	32.7521	37.9271	50.6530
		III	167.9274	196.0370	266.1923	142.3311	166.4062	226.6407	101.8844	119.4802	163.7272
	1.0	I	8.5266	9.6671	12.5024	6.8632	7.7844	10.0756	4.4276	5.0256	6.5142
		II	57.0763	65.9396	87.7265	47.7874	55.2758	73.7140	33.3535	38.6742	51.8194
		III	168.9116	197.2191	267.9051	143.1538	167.3941	228.0711	102.4581	120.1692	164.7238
	2.0	I	9.9497	11.4000	15.0616	8.0124	9.1850	12.1476	5.1730	5.9355	7.8646
		II	59.1376	68.4901	91.6712	49.4970	57.3927	76.9927	34.5252	40.1275	54.0776
		III	170.8626	199.5624	271.2998	144.7846	169.3525	230.9061	103.5952	121.5347	166.6991

Table 7.10
Values of frequency parameter for C-F annular plate for K = 0.02 and ε = 0.3

			η								
			-0.5			0.0			1.0		
			μ			μ			μ		
α	p	Mode	-0.5	0.0	1.0	-0.5	0.0	1.0	-0.5	0.0	1.0
-0.5	0.5	I	22.4321	22.6284	23.1731	18.0900	18.2495	18.6961	11.6733	11.7872	12.0970
		II	39.0422	43.6486	55.5803	32.4857	36.3796	46.4675	22.4602	25.2187	32.3810
		III	93.5288	108.9439	147.9980	78.7450	91.8614	125.1651	55.6450	65.1015	89.2236
	1.0	I	23.6353	23.8561	24.4840	19.0641	19.2433	19.7581	12.3026	12.4306	12.7873
		II	40.0099	44.6538	56.7489	33.2839	37.2103	47.4359	23.0080	25.7895	33.0475
		III	94.2297	109.7049	148.9522	79.3304	92.4977	125.9637	56.0550	65.5473	89.7833
	2.0	I	24.3491	24.6291	25.4496	19.6415	19.8697	20.5432	12.6764	12.8378	13.3016
		II	41.0500	45.8616	58.4440	34.1438	38.2099	48.8410	23.5947	26.4727	34.0107
		III	95.2715	110.9168	150.6280	80.2015	93.5114	127.3662	56.6631	66.2555	90.7640
0	0.5	I	18.6961	19.0035	19.8558	15.0529	15.3061	16.0041	9.7051	9.8778	10,3457
		II	46.4675	52.8522	68.8664	38.8111	44.2031	57.7473	26.9844	30.8086	40.4468
		III	125.1651	146.0704	198.5872	105.7371	123.5803	168.5123	75.2024	88.1512	120.9189
	1.0	I	19.7581	20.1218	21.1618	15.9103	16.2099	17.0617	10.2596	10.4637	11.0343
		II	47.4359	53.9379	70.3249	39.6143	45.1039	58.9585	27.5373	31.4288	41.2810
		III	125.9637	146.9868	199.8354	106.4057	124.3475	169.5568	75.6711	88.6888	121.6500
	2.0	I	20.5432	21.0308	22.4582	16.5466	16.9477	18.1162	10.6742	10.9458	11.7265
		II	48.8410	55.6303	72.8383	40.7783	46.5068	61.0447	28.3335	32.3898	42.7142
		III	127.3662	148.6493	202.1970	107.5788	125.7383	171.5320	76.4903	89.6602	123.0297
0.5	0.5	I	16.0041	16.4793	17.7866	12.8777	13.2650	14.3272	8.2994	8.5556	9.2535
		II	57.7473	66.1318	86.8256	48.3614	55.4515	72.9794	33.7807	38.8255	51.3393
		III	168.5123	196.5383	266.5617	142.8286	166.8332	226.9564	102.2496	119.7947	163.9612
	1.0	I	17.0617	17.6593	19.3574	13.7320	14.2191	15.5993	8.8533	9.1755	10.0827
		II	58.9585	67.5753	88.9626	49.3669	56.6502	74.7551	34.4723	39.6504	52.5628
		III	169.5568	197.7721	268.3124	143.7026	167.8651	228.4191	102.8609	120.5160	164.9817
	2.0	I	18.1162	18.9515	21.3554	14.5874	15.2681	17.2230	9.4122	9.8618	11.1473
		II	61.0447	70.1434	92.9129	51.0969	58.7814	78.0381	35.6582	41.1135	54.8238
		III	171.5320	200.1358	271.7219	145.3538	169.8408	231.2667	104.0129	121.8942	166.9663

Table 7.11

Values of frequency parameter for C-F annular plate for K = 0.0 and ε = 0.5

			η								
			-0.5			0.0			1.0		
			μ			μ			μ		
α	p	Mode	-0.5	0.0	1.0	-0.5	0.0	1.0	-0.5	0.0	1.0
-0.5	0.5	I	10.7766	12.5468	16.9775	8.5941	10.0085	13.5494	5.4535	6.3541	8.6102
		II	62.2031	74.2442	105.5493	50.8748	60.7645	86.5034	33.9588	40.6152	57.9760
		III	170.9384	205.5561	296.7110	140.8774	169.5323	245.0760	95.5095	115.1068	166.8958
	1.0	I	11.0616	12.9052	17.5461	8.8218	10.2949	14.0043	5.5984	6.5366	8.9003
		II	62.5383	74.6617	106.1998	51.1475	61.1041	87.0329	34.1389	40.8396	58.3263
		III	171.2639	205.9553	297.3127	141.1435	169.8587	245.5679	95.6872	115.3247	167.2242
	2.0	I	11.6091	13.5912	18.6247	9.2592	10.8432	14.8672	5.8769	6.8860	9.4512
		II	63.2033	75.4896	107.4888	51.6885	61.7778	88.0822	34.4962	41.2848	59.0205
		III	171.9129	206.7514	298.5126	141.6742	170.5096	246.5486	96.0415	115.7593	167.8790
0	0.5	I	13.5494	15.7532	21.2700	10.8051	12.5653	16.9729	6.8562	7.9764	10,7828
		II	86.5034	103.1015	146.1505	70.8433	84.4936	119.9339	47.4111	56.6225	80.5877
		III	245.0760	294.3972	424.0398	202.3044	243.2002	350.8259	137.6001	165.6654	239.7052
	1.0	I	14.0043	16.3273	22.1885	11.1686	13.0243	17.7076	7.0876	8.2689	11.2516
		II	87.0329	103.7642	147.1940	71.2740	85.0328	120.7835	47.6958	56.9791	81.1501
		III	245.5679	295.0015	424.9541	202.7064	243.6940	351.5728	137.8683	165.9949	240.2033
	2.0	I	14.8672	17.4110	23.9012	11.8583	13.8908	19.0781	7.5271	8.8214	12.1267
		II	88.0822	105.0768	149.2585	72.1278	86.1011	122.4646	48.2601	57.6857	82.2633
		III	246.5486	296.2066	426.7771	203.5079	244.6787	353.0619	138.4032	166.6520	241.1965
0.5	0.5	I	16.9729	19.7181	26.6027	13.5333	15.7253	21.2235	8.5852	9.9793	13.4775
		II	119.9339	142.7363	201.7382	98.3481	117.1241	165.7561	65.9854	78.6868	111.6502
		III	350.8259	420.9758	605.0369	290.0789	348.3470	501.4188	197.9583	238.0867	343.7650
	1.0	I	17.7076	20.6497	28.1072	14.1205	16.4700	22.4267	8.9592	10.4538	14.2450
		II	120.7835	143.8055	203.4410	99.0395	117.9945	167.1430	66.4427	79.2628	112.5690
		III	351.5728	421.8952	606.4336	290.6888	349.0977	502.5586	198.3650	238.5871	344.5240
	2.0	I	19.0781	22.3752	30.8475	15.2161	17.8498	24.6195	9.6575	11.3338	15.6451
		II	122.4646	145.9197	206.8025	100.4078	119.7158	169.8814	67.3479	80.4022	114.3841
		III	353.0619	423.7283	609.2179	291.9051	350.5946	504.8311	199.1758	239.5848	346.0374

Table 7.12
Values of frequency parameter for C-F annular plate for K = 0.02 and ε = 0.5

α	p	Mode	η								
			-0.5			0.0			1.0		
			μ			μ			μ		
			-0.5	0.0	1.0	-0.5	0.0	1.0	-0.5	0.0	1.0
-0.5	0.5	I	24.5563	25.3753	27.8191	19.5884	20.2454	22.2041	12.4302	12.8534	14.1099
		II	65.6659	77.1606	107.6095	53.6958	63.1420	88.1852	35.8415	42.2041	59.1030
		III	172.1921	206.5968	297.4288	141.9051	170.3860	245.6656	96.2061	115.6863	167.2972
	1.0	I	25.7398	26.5759	29.0976	20.5340	21.2049	23.2264	13.0313	13.4638	14.7615
		II	66.3515	77.8748	108.4705	54.2536	63.7233	88.8864	36.2118	42.5899	59.5682
		III	172.6521	207.1078	298.1076	142.2814	170.8040	246.2208	96.4585	115.9665	167.6687
	2.0	I	26.4848	27.4025	30.1995	21.1304	21.8669	24.1094	13.4118	13.8867	15.3265
		II	67.1610	78.8238	109.8434	54.9119	64.4953	90.0040	36.6472	43.1007	60.3081
		III	173.3643	207.9563	299.3436	142.8638	171.4978	247.2312	96.8479	116.4302	168.3437
0	0.5	I	22.2041	23.6107	27.5945	17.7074	18.8331	22.0199	11.2347	11.9544	13.9887
		II	88.1852	104.5146	147.1482	72.2189	85.6502	120.7516	48.3355	57.4009	81.1394
		III	245.6656	294.8874	424.3793	202.7902	243.6044	351.1063	137.9327	165.9427	239.8981
	1.0	I	23.2264	24.6936	28.9003	18.5239	19.6985	23.0642	11.7541	12.5053	14.6547
		II	88.8864	105.3215	148.2931	72.7900	86.3075	121.6843	48.7144	57.8368	81.7579
		III	246.2208	295.5444	425.3301	203.2444	244.1417	351.8833	138.2367	166.3019	240.4169
	2.0	I	24.1094	25.7527	30.5120	19.2307	20.5464	24.3552	12.2053	13.0470	15.4803
		II	90.0040	106.6908	150.3962	73.6995	87.4220	123.3969	49.3161	58.5743	82.8922
		III	247.2312	296.7741	427.1700	204.0702	245.1466	353.3863	138.7882	166.9728	241.4197
0.5	0.5	I	22.0199	24.1992	30.0750	17.5575	19.2989	23.9935	11.1373	12.2465	15.2362
		II	120.7516	143.4240	202.2253	99.0194	117.6891	166.1569	66.4393	79.0694	111.9223
		III	351.1063	421.2094	605.1995	290.3111	348.5407	501.5538	198.1189	238.2209	343.8588
	1.0	I	23.0642	25.3931	31.7555	18.3918	20.2531	25.3376	11.6684	12.8544	16.0935
		II	121.6843	144.5628	203.9770	99.7789	118.6166	167.5839	66.9426	79.6840	112.8684
		III	351.8833	422.1539	606.6136	290.9460	349.3122	502.7081	198.5428	238.7356	344.6279
	2.0	I	24.3552	27.0157	34.3624	19.4248	21.5516	27.4246	12.3276	13.6835	17.4272
		II	123.3969	146.7029	207.3559	101.1730	120.3592	170.3365	67.8652	80.8378	114.6931
		III	353.3863	423.9987	609.4060	292.1738	350.8187	504.9872	199.3616	239.7400	346.1459

Table 7.13

Comparison of frequency parameter Ω for isotropic/polar orthotropic homogeneous ($\mu = 0.0$, $\eta = 0.0$) annular plates of uniform thickness ($\alpha = 0.0$) for $\varepsilon = 0.3$, $\upsilon_\theta = 0.3$

Edge Conds.	Mode	p = 1				p = 5			
		K = 0.0		K = 0.01		K = 0.0		K = 0.01	
C-C	I	45.3462	45.3371*	46.5347	46.5259*	48.3540	48.3321*	49.5575	49.5364*
	II	125.3621	125.6191*	125.7969	126.0531*	129.6030	129.8250*	130.0568	130.2778*
	III	246.1573	246.6944*	246.3790	246.9155*	250.9706	251.4816*	251.2052	251.7157*
C-S	I	29.9777	29.9689*	31.7469	31.7385*	33.2692	33.2528*	34.9954	34.9798*
	II	100.4228	100.6065*	100.9651	101.1478*	104.7739	104.9319*	105.3348	105.4919*
	III	211.1294	211.5629*	211.3878	211.8208*	216.0447	216.4574*	216.3172	216.7293*
C-F	I	6.6604	6.6542*	12.3920	12.3880*	9.9163	9.9073*	14.7028	14.6967*
	II	42.6142	42.6156*	43.8768	43.8782*	47.8208	47.8100*	49.0374	49.0268*
	III	123.4661	123.5739*	123.9076	124.0152*	128.8027	128.8986*	129.2593	129.3546*

* Values taken from Verma[1987].

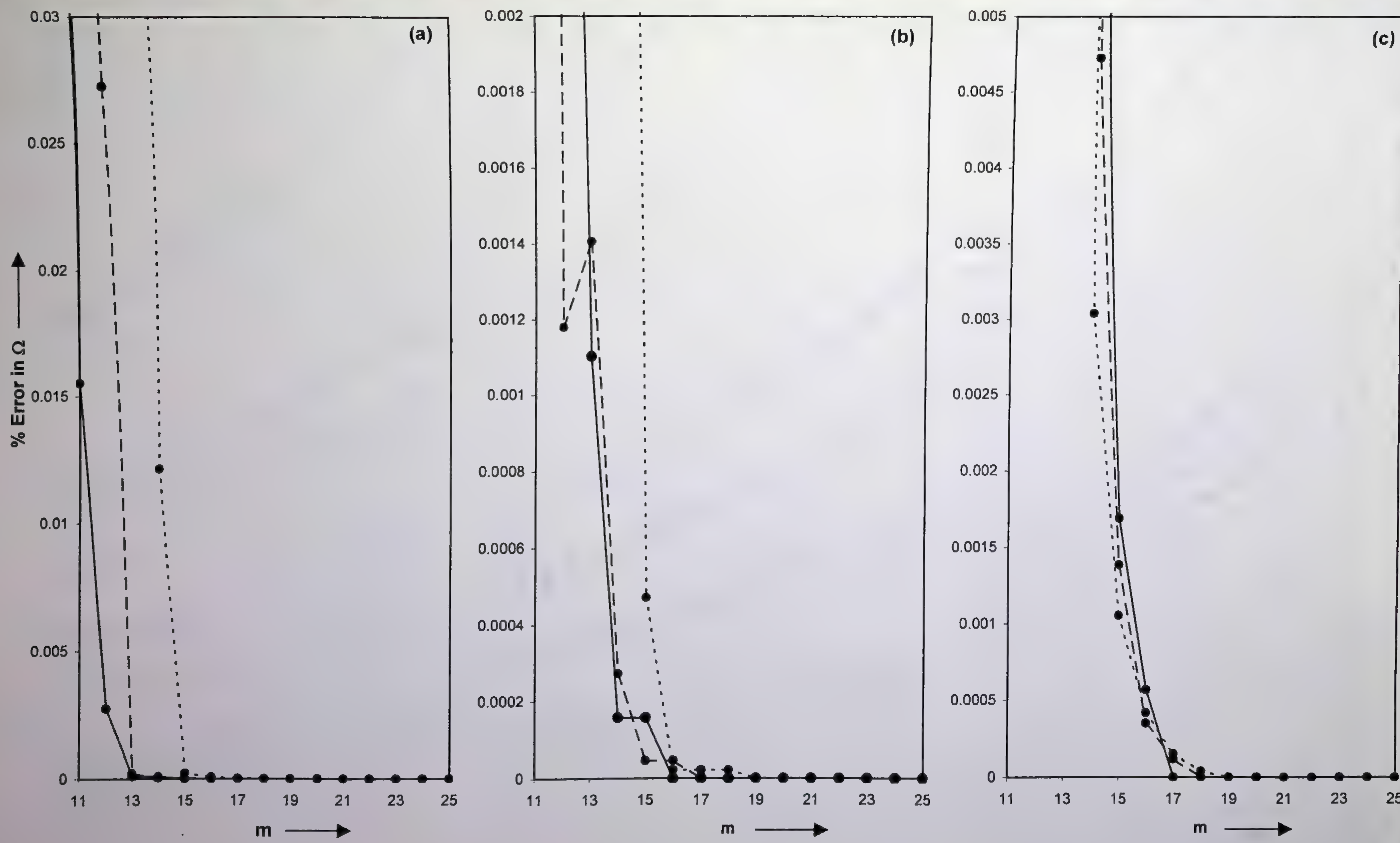

Fig. 7.1 : Convergence for first three modes of vibration for (a) C-C (b) C-S (c) C-F plate for ε=0.3, μ=1.0, η=-0.5, α=0.5, K=0.02, p=2.0.
———, Fundamental mode; – – – – – –, Second mode; ------------, Third mode.
Percentage error= $\left(\frac{|\Omega_m - \Omega_{19}|}{\Omega_{19}}\right) \times 100$

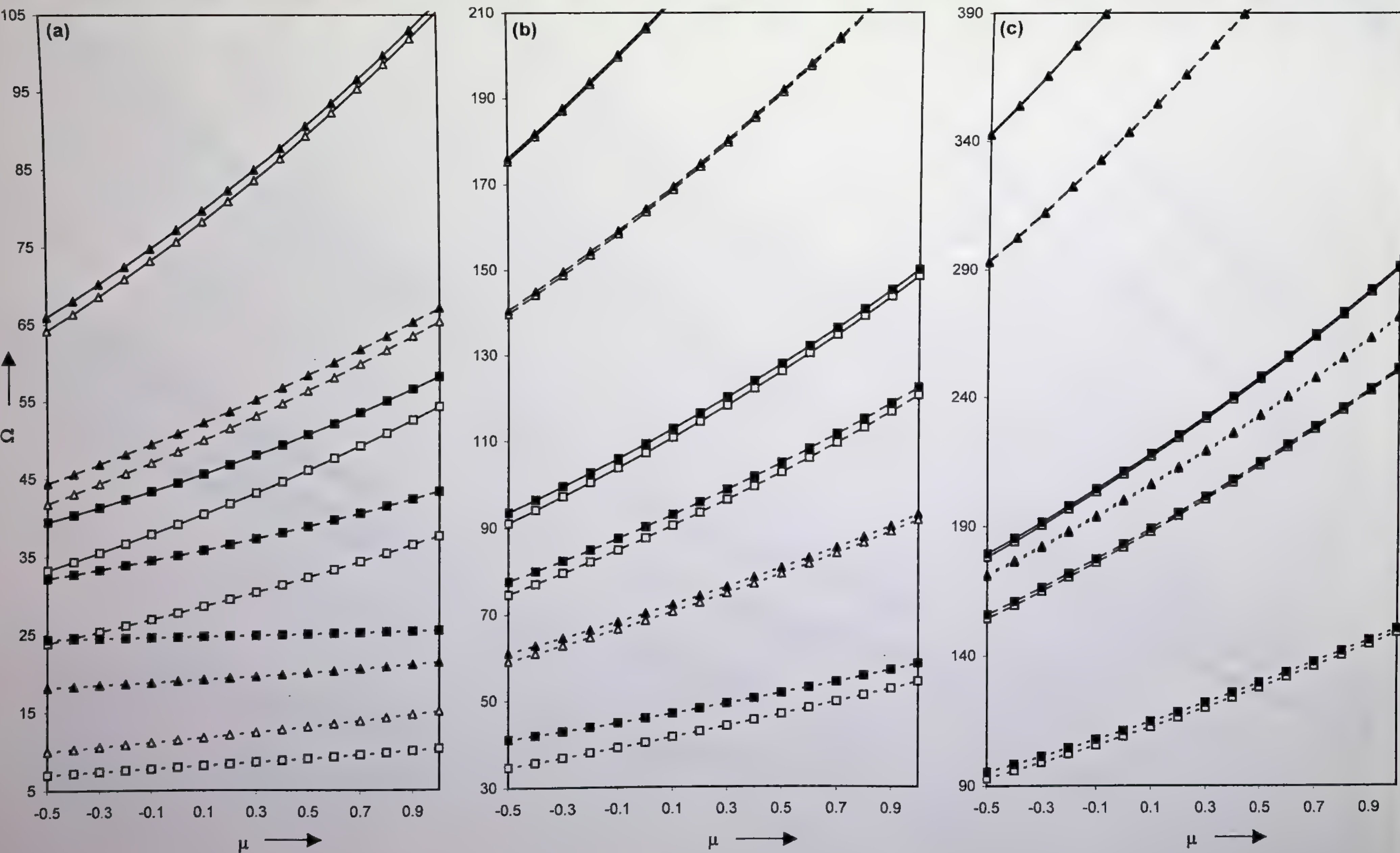

Fig. 7.2 : Frequency parameter for C-C, C-S and C-F plates for (a) fundamental (b) second and (c) third mode for ε = 0.3, η = -0.5, p = 2.0.
———, C-C; – – – – –, C-S; ----------,C-F.
□, α = -0.5; Δ, α = 0.5. □, Δ, K = 0.0; ■, ▲, K = 0.02.

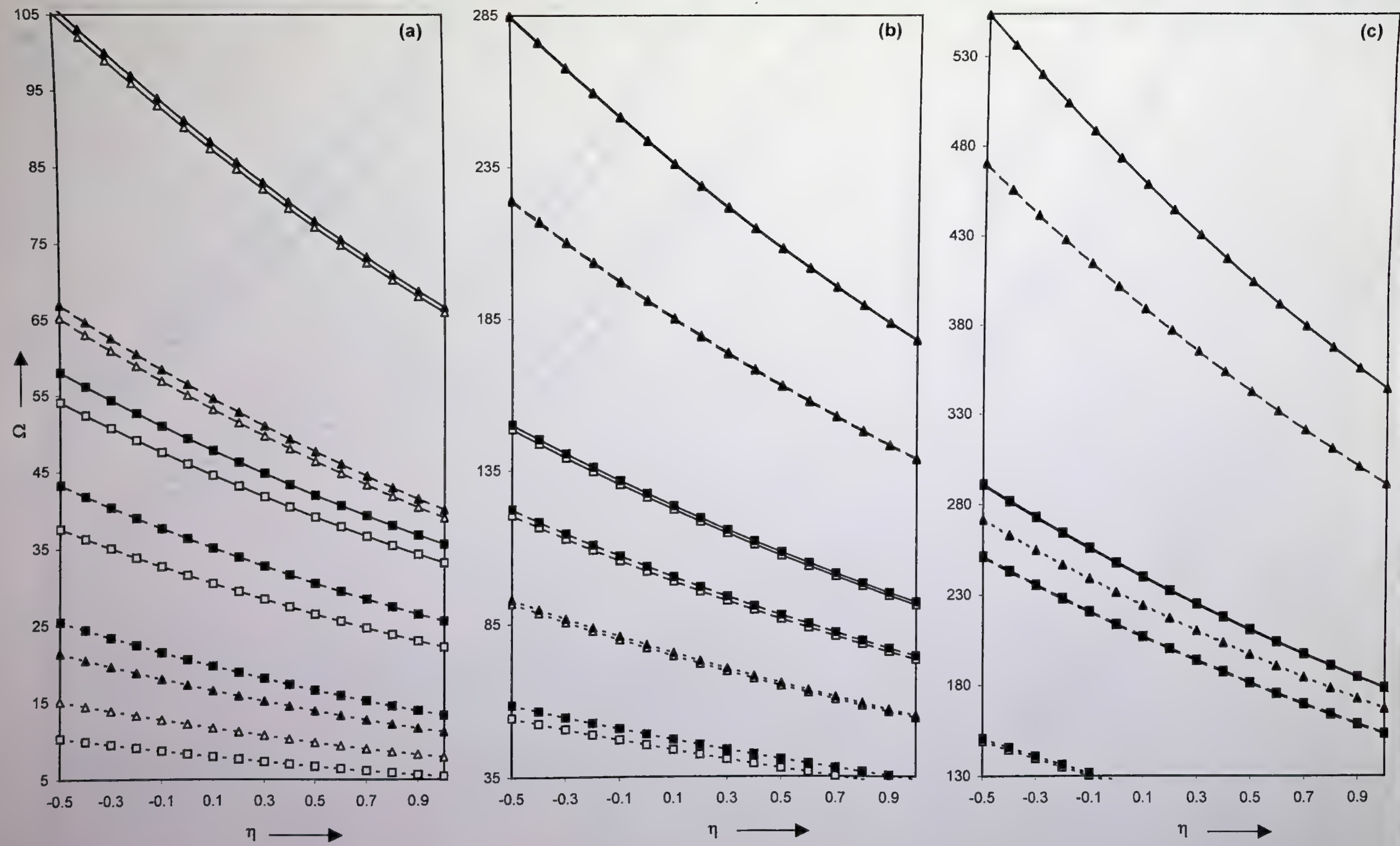

Fig. 7.3 : Frequency parameter for C-C, C-S and C-F plates for (a) fundamental (b) second and (c) third mode for $\varepsilon = 0.3$, $\mu = 1.0$, $p = 2.0$.
———, C-C; – – – – –, C-S; ----------,C-F.
□, $\alpha = -0.5$; Δ, $\alpha = 0.5$. □, Δ, K = 0.0; ■, ▲, K = 0.02.

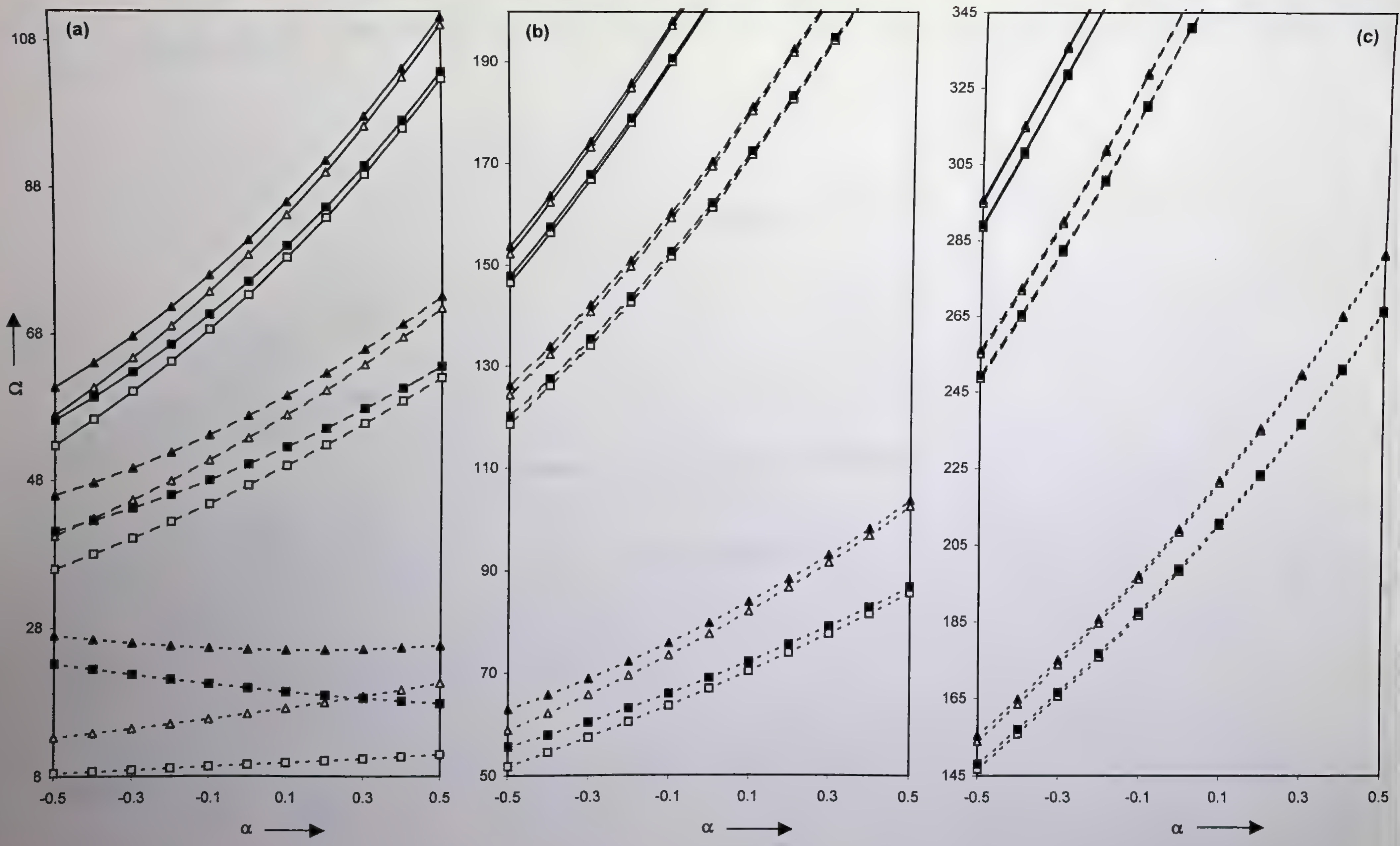

Fig. 7.4 : Frequency parameter for C-C, C-S and C-F plates for (a) fundamental (b) second and (c) third mode for $\varepsilon = 0.3$, $\mu = 1.0$, $\eta = -0.5$.
———, C-C; – – – – –, C-S; ----------, C-F.
□, $p = 0.5$; Δ, $p = 5.0$. □, Δ, $K = 0.0$; ■, ▲, $K = 0.02$.

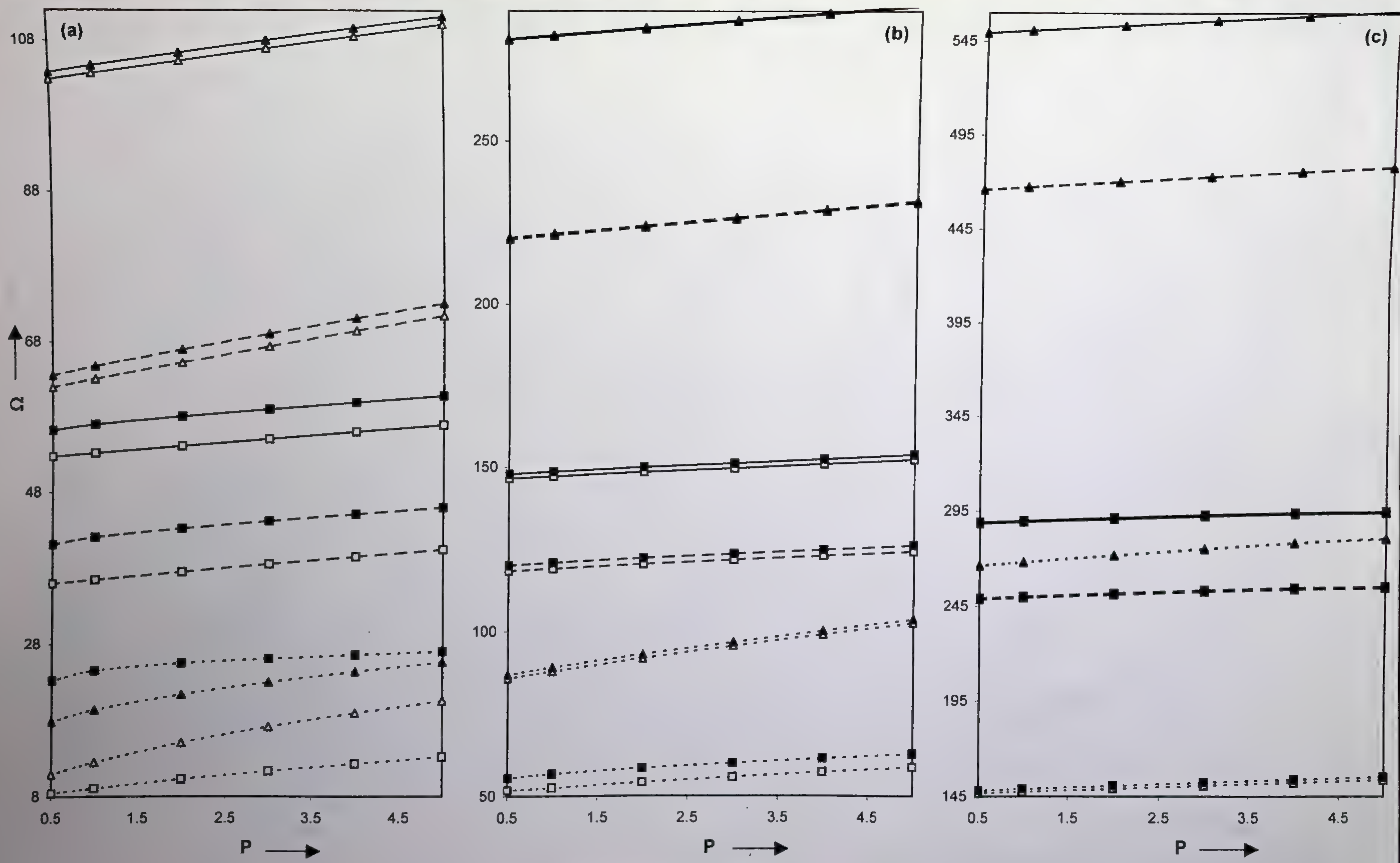

Fig. 7.5 : Frequency parameter for C-C, C-S and C-F plates for (a) fundamental (b) second and (c) third mode for $\varepsilon = 0.3$, $\mu = 1.0$, $\eta = -0.5$.
———, C-C; – – – – –, C-S; ----------, C-F.
□, $\alpha = -0.5$; Δ, $\alpha = 0.5$. □, Δ, K = 0.0; ■, ▲, K = 0.02.

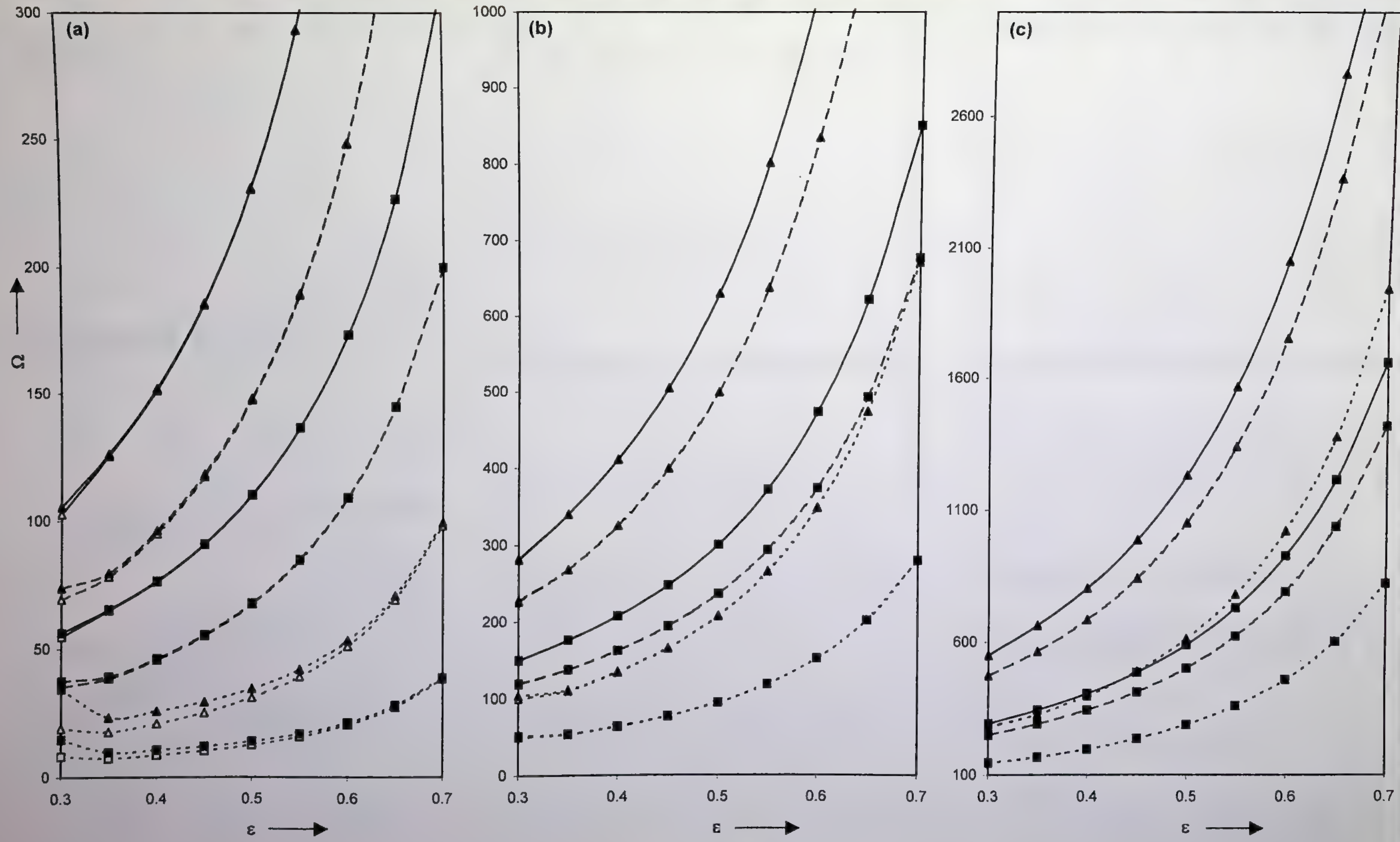

Fig. 7.6 : Frequency parameter for C-C, C-S and C-F plates for (a) fundamental (b) second and (c) third mode for $\mu = 1.0$, $\eta = -0.5$, $p = 2.0$.

———, C-C; – – – –, C-S; ----------, C-F.

□, $\alpha = -0.5$; Δ, $\alpha = 0.5$. □, Δ, $K = 0.0$; ■, ▲, $K = 0.02$.

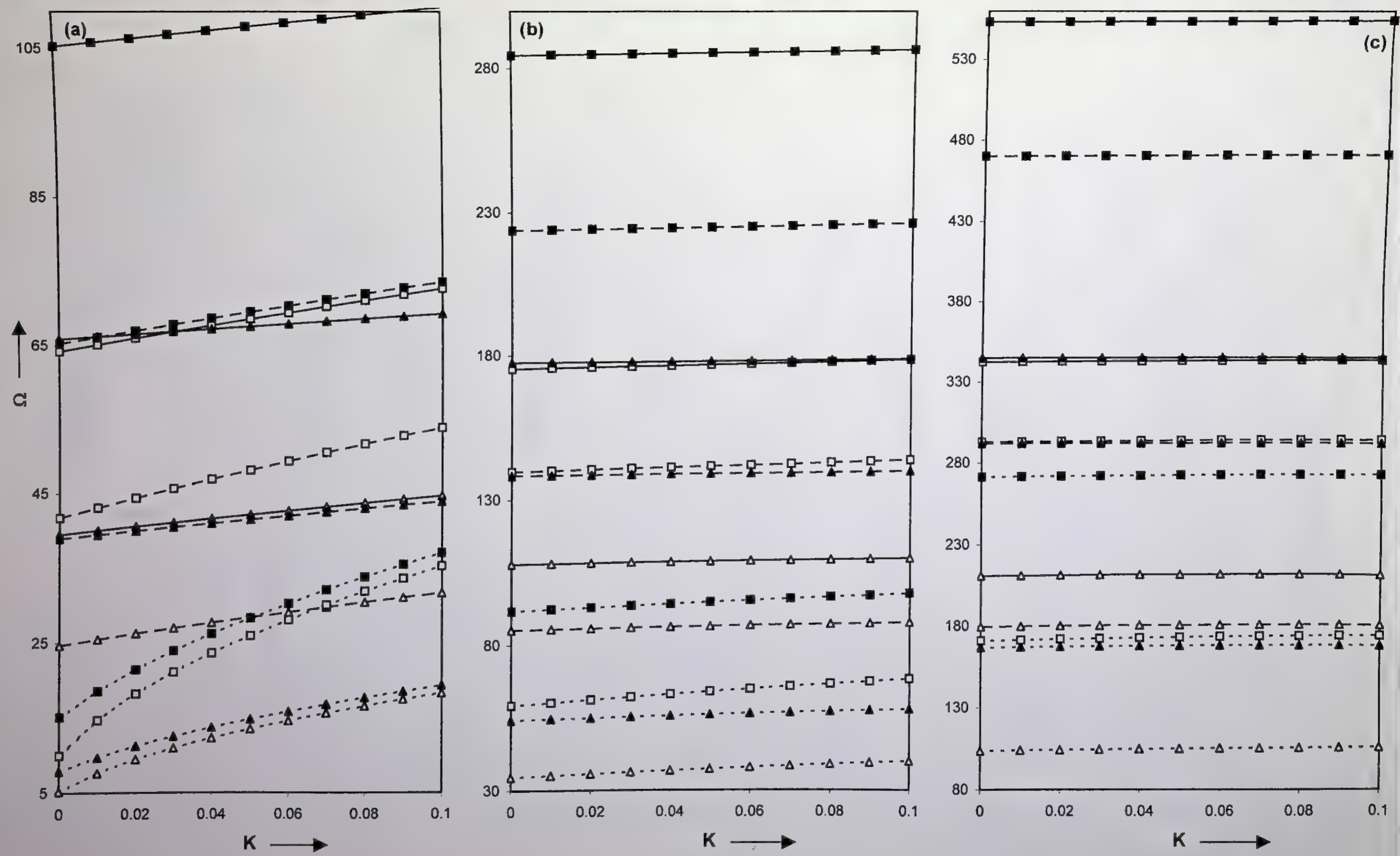

Fig. 7.7 : Frequency parameter for C-C, C-S and C-F plates for (a) fundamental (b) second and (c) third mode for $\varepsilon = 0.3$, $p = 2.0$, $\alpha = 0.5$.
———, C-C; – – – – –, C-S; -----------, C-F.
□, $\eta = -0.5$; Δ, $\eta = 1.0$. □, Δ, $\mu = -0.5$; ■, ▲, $\mu = 1.0$.

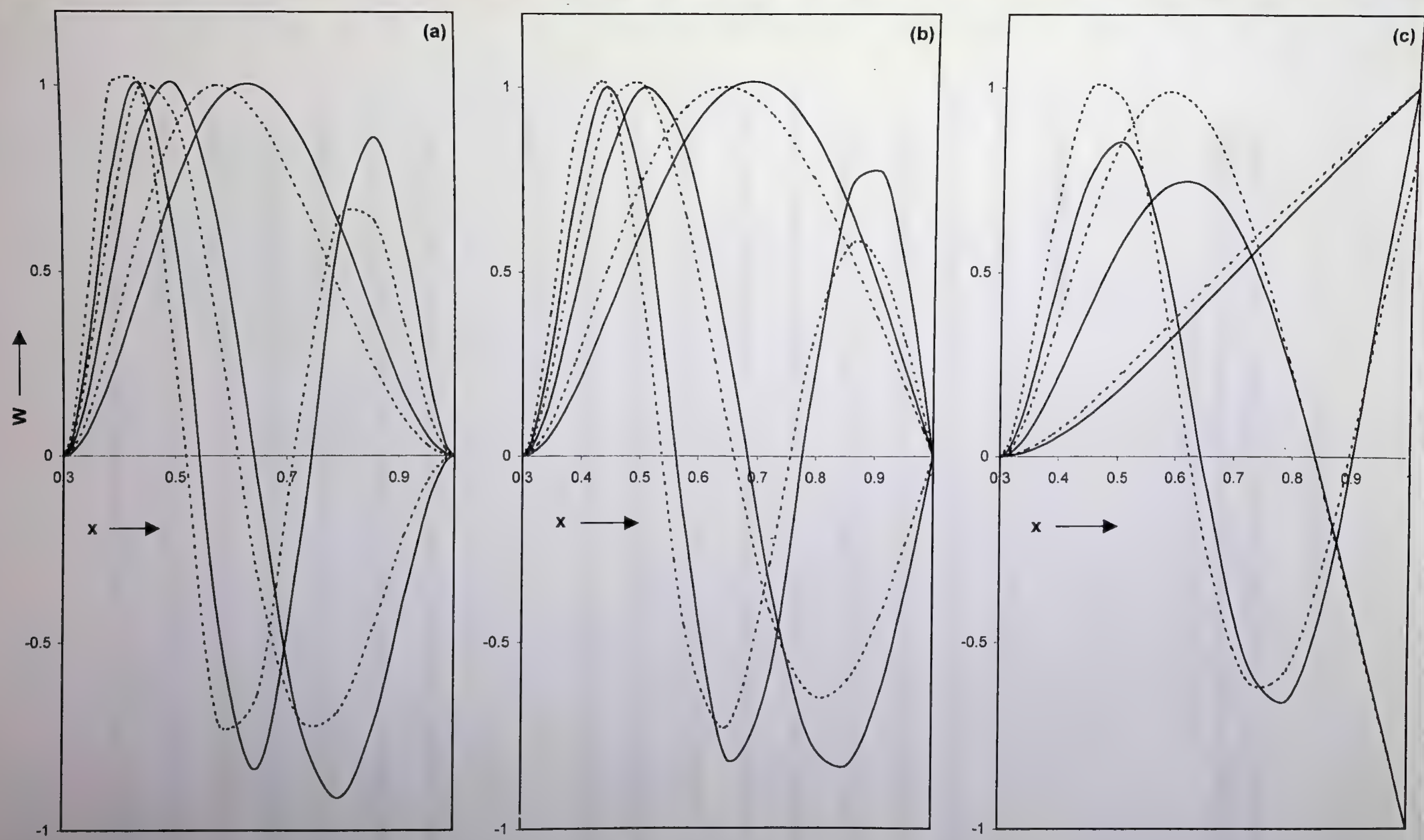

Fig. 7.8 : Normalized displacements for first three modes of vibration for (a) C-C (b) C-S and (c) C-F plate for $\varepsilon = 0.3$, $\mu = 1.0$, $\eta = -0.5$, $p = 2.0$, $K = 0.02$. ———, $\alpha = -0.5$; ----------, $\alpha = 0.5$.

CHAPTER VIII

EFFECT OF PASTERNAK FOUNDATION ON AXISYMMETRIC VIBRATION OF NON-HOMOGENEOUS NON-UNIFORM POLAR ORTHOTROPIC ANNULAR PLATE

1. INTRODUCTION

Due to the desirability of lightweight and high performance characteristics, circular/annular plates of composite material are being widely used in various engineering applications. Plates on elastic foundation are of great practical interest in connection with reinforced concrete pavements of highways, airport runways, building footings, design of storage tanks, deep sea pressure vessels and machine bases etc. To fill up the important deficiency regarding discontinuity of displacement in Winkler model, various other models such as Hetenyi. Reissner, Vlasov and Pasternak etc. have been proposed in the literature. An excellent discussion on foundation models is given by Kerr[1964]. Of these, Pasternak model provides a better approximation as it takes into account, not only its transverse reaction but also the shear interaction between spring elements.

In the recent past, vibration problems with regard to plates having uniform / variable thickness resting on Pasternak foundation have attracted research workers due to their important role in foundation engineering and also due to the availability of various numerical techniques and computational facilities. Wang et al.[1997] obtained natural frequencies of isotropic Reddy plates on a Pasternak foundation. Omurtag and Kadioglu[1998] carried out free vibration analysis of orthotropic Kirchhoff rectangular plates resting on Pasternak foundation by finite element method. Shen et al.[2001] presented free and forced vibration analysis of moderately

thick isotropic rectangular plates with free edges resting on Pasternak type elastic foundation employing Rayleigh-Ritz method. Malekzadeh and Karami[2004] obtained a differential quadrature method(DQM) solution for free vibration analysis of isotropic non-uniform thick plates resting on Pasternak foundation. Zhou et al.[2006] in their recent paper, presented three dimensional free vibration analysis of thick isotropic circular plates on Pasternak foundation. It shows that no work has been done to study the effect of Pasternak foundation on the natural frequencies of polar orthotropic annular plates of variable thickness.

In this chapter, axisymmetric vibrations of non-homogeneous polar orthotropic annular plates of variable thickness resting on a Pasternak type elastic foundation have been studied on the basis of classical plate theory. Hamilton's energy principle has been used to derive the governing differential equation of motion. Frequency equations for an annular plate for two different combinations of edge conditions have been obtained employing Chebyshev collocation technique. Numerical results, thus obtained, have been presented in the form of tables and graphs. The effect of foundation parameters and thickness variation together with various plate parameters such as rigidity ratio, radii ratio, taper parameter on natural frequencies has been investigated for the first three modes of vibration. Mode shapes for specified plates have also been presented.

2. EQUATION OF MOTION

Consider a non-homogeneous polar orthotropic annular plate of inner and outer peripheral radii b and a, respectively, thickness $h(r)$ and density $\rho(r)$ resting on Pasternak foundation with spring and shear stiffness K_f and G_f, respectively, referred to cylindrical polar coordinate system (r, θ, z).

Energy Variations

The work done by the foundation is given by

$$W_{foundation} = \frac{1}{2}\int_b^a \int_0^{2\pi} \left(K_f w^2 + G_f \left(\frac{\partial w}{\partial r} \right)^2 \right) r \, dr \, d\theta \quad . \tag{8.2.1}$$

Applying Hamilton's energy principle as in chapter VII and substituting $W_{foundation}$ (as given by relation (8.2.1)) in equations (7.2.2)-(7.2.9), small deflection axisymmetric motion of such a plate is governed by the equation

$$\begin{aligned} D_r w_{,rrrr} + 2\frac{(D_r + rD_{r,r})}{r} w_{,rrr} + \frac{(-D_\theta + (2+\nu_\theta) r D_{r,r} + r^2 D_{r,rr} - r^2 G_f)}{r^2} w_{,rr} \\ + \frac{(D_\theta - rD_{\theta,r} + r^2 \nu_\theta D_{r,rr} - r^2 G_f)}{r^3} w_{,r} + K_f w + \rho h w_{,tt} = 0, \end{aligned} \tag{8.2.2}$$

where a comma followed by a suffix represents the partial differentiation with respect to that variable and $(D_r, D_\theta) = \frac{(E_r, E_\theta) h^3}{12(1-\upsilon_r \upsilon_\theta)}$ are the flexural rigidities of the plate, w the transverse deflection, t the time, E_r, E_θ, υ_r, υ_θ are respectively the Young's moduli and Poisson's ratios of the plate material in the proper directions with $\upsilon_r E_\theta = E_r \upsilon_\theta$.

For a non-homogeneous plate, equation (8.2.2) reduces to

$$\begin{aligned} & E_r \frac{\partial^4 w}{\partial r^4} + \frac{2}{r}\left[E_r + r\frac{dE_r}{dr} \right] \frac{\partial^3 w}{\partial r^3} \\ & + \frac{1}{r^2}\left[-E_\theta + r(2+\upsilon_\theta)\frac{dE_r}{dr} + r^2 \frac{d^2 E_r}{dr^2} - \frac{12(1-\upsilon_r\upsilon_\theta)}{h^3} r^2 G_f \right] \frac{\partial^2 w}{\partial r^2} \\ & + \frac{1}{r^3}\left[E_\theta - r\frac{dE_\theta}{dr} + r^2 \upsilon_\theta \frac{d^2 E_r}{dr^2} - \frac{12(1-\upsilon_r\upsilon_\theta)}{h^3} r^2 G_f \right] \frac{\partial w}{\partial r} \\ & + \frac{12(1-\upsilon_r\upsilon_\theta)}{h^3} K_f w + \frac{12(1-\upsilon_r\upsilon_\theta)\rho}{h^2} \frac{\partial^2 w}{\partial t^2} = 0. \end{aligned} \tag{8.2.3}$$

Introducing the non-dimensional variables $x=\frac{r}{a}, \bar{w}=\frac{w}{a}, \bar{h}=\frac{h}{a}$ together with general thickness variation along radial direction, i.e.

$$\bar{h}=h_0(1+\alpha x^n), \tag{8.2.4}$$

and exponential variation along radial direction in Young's moduli and density to account for non-homogeneity of plate material, i.e.

$$E_r=E_1e^{\mu x}, \qquad E_\theta=E_2e^{\mu x}, \qquad \rho=\rho_0e^{\eta x}, \tag{8.2.5}$$

equation (8.2.3) reduces to

$$P_0\frac{d^4W}{dx^4}+P_1\frac{d^3W}{dx^3}+P_2\frac{d^2W}{dx^2}+P_3\frac{dW}{dx}+P_4W=0\ , \tag{8.2.6}$$

where, $\bar{w}(x,t)=W(x)e^{i\omega t}$ (for harmonic vibrations), ω is the radian frequency, E_1, E_2 are Young's moduli and h_0 is thickness of plate at the centre, α is the taper parameter,

$$P_0=H^3\ , \qquad P_1=\frac{2}{x}\left[H^3+AH^2x\right],$$

$$P_2=\frac{1}{x^2}\left[-pH^3+(2+\upsilon_\theta)AH^2x+\left(BH-Ge^{-\mu x}\right)x^2\right],$$

$$P_3=\frac{1}{x^3}\left[p\left(H^3-AH^2x\right)+\left(\upsilon_\theta BH-Ge^{-\mu x}\right)x^2\right],$$

$$P_4=K^*e^{-\mu x}-\Omega^2He^{(\eta-\mu)x}\ ,$$

$$A=\mu H+3H', \qquad B=3\left(HH''-H'^2\right)+A^2\ , \qquad p=\frac{E_2}{E_1}\ ,$$

$$K^*=\frac{aK_f}{D_1}\ , \qquad G=\frac{G_f}{aD_1}\ , \qquad D_1=\frac{E_1h_0^3}{12(1-\upsilon_r\upsilon_\theta)}\ , \qquad H=1+\alpha x^n\ ,$$

$$\Omega^2=\frac{12\rho_0a^2\omega^2(1-\upsilon_r\upsilon_\theta)}{E_1h_0^2}\ ,$$

and ρ_0 is density at $x=0$.

Equation (8.2.6) together with boundary conditions at the edges $x = \varepsilon$ and $x = 1$, where $\varepsilon = b/a$, constitutes a two point boundary value problem in the range $(\varepsilon, 1)$, which has been solved by Chebyshev collocation technique.

3. METHOD OF SOLUTION : CHEBYSHEV COLLOCATION TECHNIQUE

The range of the plate, namely $\varepsilon \le x \le 1$ is transformed to $-1 \le y \le 1$, which is the applicability range of the Chebyshev collocation technique by choosing a new independent variable

$$y \equiv \frac{1}{(1-\varepsilon)}\{2x-(1+\varepsilon)\} \tag{8.3.1}$$

and equation (8.2.6) reduces to

$$A_0\frac{d^4W}{dy^4}+A_1\frac{d^3W}{dy^3}+A_2\frac{d^2W}{dy^2}+A_3\frac{dW}{dy}+A_4W=0, \tag{8.3.2}$$

where, $A_i = \xi^{4-i}P_i, i = 0,1,2,3,4$ and $\xi = 2/(1-\varepsilon)$. According to Chebyshev Collocation technique, assuming the highest order derivative of W, as the linear sum of Chebyshev polynomials (equation (7.3.3)),

$$W = c_1 + c_2T_1 + c_3T_1^1 + c_4T_1^2 + \sum_{k=0}^{m-5} c_{k+5}T_k^4, \tag{8.3.3}$$

where, c_1, c_2, c_3 and c_4 are the constants of integration and T_k^j represents the j^{th} integral of T_k.

Substitution of W and its derivatives in equation (8.3.2) gives an equation in terms of the T's and c's. Satisfaction of this resultant equation at (m-4) collocation points given by

$$y_i = \cos\left(\frac{2i-1}{m-4}\cdot\frac{\pi}{2}\right), \quad i = 1,2,...,m-4 \tag{8.3.4}$$

provides a set of (m-4) equations in unknowns a_j ($j = 1, 2,\ldots, m$), which can be written in

matrix form as

$$[B][C^{*}]=[0] \ , \tag{8.3.5}$$

where, B and C^{*} are matrices of order $(m\text{-}4)$ x m and m x 1, respectively.

4. BOUNDARY CONDITIONS AND FREQUENCY EQUATIONS

By satisfying the relations

$$W=\frac{dW}{dy}=0, \qquad \text{for clamped edge and}$$

$$W=\xi\frac{d^{2}W}{dy^{2}}+\frac{\upsilon_{\theta}}{x}\frac{dW}{dy}=0 \ , \qquad \text{for simply supported edge}$$

a set of four homogeneous equations is obtained for (i) C-C (both the inner and outer edges clamped), (ii) C-S (clamped at the inner edge and simply supported at the outer). These equations together with the field equations (8.3.5) give a complete set of m equations in m unknowns, which for a C-C plate can be written as

$$\begin{bmatrix} B \\ B^{cc} \end{bmatrix}[C^{*}]=[0] \ , \tag{8.4.1}$$

where B^{cc} is a matrix of order 4 x m.

For a non-trivial solution of equation (8.4.1), the frequency determinant must vanish and hence

$$\begin{vmatrix} B \\ B^{cc} \end{vmatrix}=0 \cdot \tag{8.4.2}$$

Similarly for C-S plate, frequency determinant can be written as

$$\begin{vmatrix} B \\ B^{cs} \end{vmatrix}=0. \tag{8.4.3}$$

5. NUMERICAL RESULTS AND DISCUSSION

The frequency equations (8.4.2) and (8.4.3) have been solved to obtain the values of the frequency parameter Ω for various values of plate parameters. In order to investigate the effect of non-homogeneity parameter μ = -0.5(0.1)1.0, density parameter η = -0.5(0.1)1.0, taper parameter α = -0.5(0.1)0.5, rigidity parameter p = 0.5, 1.0, 2.0, 5.0, radii ratio ε = 0.3(0.05)0.5 on natural frequencies of C-C and C-S plates resting on Pasternak foundation with stiffness parameters K^* = 0(100)500 and G = 0(5)25 for $\upsilon_\theta = 0.3$, numerical results have been computed for the first three modes of vibration for Linearly Varying Thickness (LVT) and Parabolically Varying Thickness (PVT) plates.

To choose appropriate value of the number of collocation points m, convergence study was carried out for annular plates for different sets of plate parameters. Convergence graphs for C-C and C-S plates for LVT are shown in Figures 8.1(a,b) respectively for $\mu = 1.0$, $\eta = -0.5$, $p = 0.5$, $\varepsilon = 0.5$, $\alpha = 0.5$, $K^* = 200$, $G = 25$. It is observed that $m = 18$ gives first three frequency parameters with at least four decimal accuracy.

The results are presented in Tables (8.1-8.12) and Figures (8.2-8.9). Tables (8.1-8.12) present the first three frequency parameters Ω of axisymmetric vibration for various values of plate parameters i.e. p = 0.5, 1.0, 2.0, α = -0.5, 0.0, 0.5, K^* = 0, 200, G = 0, 10, 25 for three sets of non-homogeneity and density parameters $\mu = -0.5$, $\eta = 1.0$; $\mu = 1.0$, $\eta = -0.5$; $\mu = 0.0$, $\eta = 0.0$ and ε = 0.3 and 0.5 for LVT as well as PVT plates for C-C and C-S boundary conditions. From these results, it is found that the values of frequency parameters Ω are higher for LVT plate than those for PVT Plate for positive values of taper constant α keeping all other plate parameters constant, while the behaviour of Ω is just the reverse for negative value of α.

Figures 8.2(a,b,c) show the behaviour of frequency parameter Ω with non-homogeneity parameter μ for η = -0.5, α = 0.5, ε = 0.3, p = 5.0 for both LVT and PVT plates vibrating in fundamental, second and third mode, respectively. Three groups of different foundation stiffness parameters, namely K^* = 0, G = 0; K^* = 500, G = 0 and K^* = 500, G = 25 have been considered for C-C and C-S plates. It is found that frequency parameter Ω increases with increasing value of μ. The frequency parameter Ω is found to increase with increasing value of foundation parameter K^* (Winkler foundation stiffness) and also with that of parameter G (shear stiffness), for both C-C and C-S plates. The effect of foundation decreases with increasing value of non-homogeneity parameter μ. The effect of K^* on Ω decreases, while that of G increases with increasing order of modes. However, overall effect of Pasternak foundation is found to increase with increase in the number of modes.

Figures 8.3(a,b,c) show the plots of first three frequency parameters Ω versus density parameter η for μ = 1.0, α = 0.5, ε = 0.3, p = 5.0 and K^* = 0, G = 0; K^* = 500, G = 0; K^* = 500, G = 25 for both LVT and PVT plates. The frequency is found to decrease with increasing value of density parameter η for both the plates. The rate of decrease of Ω increases with increase in the number of modes.

Figures 8.4(a,b,c) show the effect of taper parameter α on frequency parameter Ω for μ = 1.0, η = -0.5, ε = 0.3, p = 5.0 for both LVT and PVT plates vibrating in fundamental, second and third mode respectively. The frequency parameter Ω is found to increase with increasing value of the taper parameter α except for C-S plate with K^* = 500, G = 25. In this case, Ω first decreases and then increases with a local minima in vicinity of α = -0.4. It is observed that Ω_{LVT} < Ω_{PVT} for α < 0 while Ω_{LVT} > Ω_{PVT} for α > 0. The rate of increase of Ω with α, becomes more

pronounced with increase in the number of modes. The effect of foundation parameter on Ω is found to decrease with increasing value of taper parameter α.

Figure 8.5a depicts the variation of frequency parameter Ω with rigidity parameter p for $\mu = 1.0$, $\eta = -0.5$, $\varepsilon = 0.3$, $\alpha = 0.5$, $K^* = 0$, $G = 0$; $K^* = 500$, $G = 0$; $K^* = 500$, $G = 25$ for both LVT and PVT plates vibrating in fundamental mode. The frequency is found to increase as the plate becomes more and more stiff in the tangential direction as compared to that in radial direction. The rate of increase of frequency parameter with increase in the value of p for LVT plate, is slightly higher than that for PVT plate. A similar inference is drawn in the case of the plate vibrating in second and third mode (Figures 8.5(b,c)).

Figure 8.6 shows the behaviour of frequency parameter Ω with radii ratio ε for $\mu = 1.0$, $\eta = -0.5$, $\alpha = 0.5$, p = 1.0, 5.0, $K^* = 500$ and G = 25 for fundamental mode of vibration for C-C and C-S plate. It is found that frequency parameter increases with increasing value of radii ratio. The rate of increase is more pronounced for large value of radii ratio ($\varepsilon \geq 0.5$). The effect of rigidity decreases with increasing value of radii ratio.

Figures 8.7(a,b,c) depict the effect of foundation parameter K^* on frequency parameter Ω for $\mu = 1.0$, $\eta = -0.5$, $\varepsilon = 0.3$, $p = 5.0$, $\alpha = 0.5$ and G = 0, 25 for LVT and PVT plates vibrating in fundamental, second and third mode respectively. It is observed that frequency parameter increases linearly with increasing value of K^*. The rate of increase in the value of Ω with increase in K^* for PVT plate is slightly more than that for LVT plate. The rate of increase decreases with the increase in the number of modes for both LVT and PVT plates.

Figures 8.8(a,b,c) show the effect of shear stiffness foundation parameter G on frequency parameter Ω for $\mu = 1.0$, $\eta = -0.5$, $\varepsilon = 0.3$, $p = 5.0$, $\alpha = 0.5$ and $K^* = 0, 500$ for LVT and PVT plates vibrating in fundamental, second and third mode, respectively. The frequency parameter is found to increase linearly with increasing value of G. The rate of increase in the value of Ω with increase in G for PVT plate is slightly more than that for LVT plate. The rate of increase increases with the increase in the number of modes for both LVT and PVT plates. The natural frequencies obtained by Pasternak foundation model are higher than that for Winkler model.

Figures 8.9(a,b) show the plots for normalised transverse displacements for $\mu = 1.0$, $\eta = -0.5$, $\varepsilon = 0.3$, $p = 5.0$, $\alpha = -0.5, 0.5$, $K^* = 200$ and $G = 25$ both for LVT and PVT plates for the first three modes of vibration for C-C and C-S plates respectively. The radii of the nodal circles decrease as the outer edge becomes thicker and thicker for both LVT and PVT plates.

Table 8.13 shows the comparison of results for isotropic ($p = 1.0$)/polar orthotropic ($p = 5.0$) annular plates of uniform thickness($\alpha = 0.0$) resting on Winkler foundation($K = 0.01$, $K = 0.02$; $G = 0.0$), where $K = K^* h_0^3 / 12\left(1 - \frac{\nu_\theta^2}{p}\right)$, with solutions obtained by Verma[1987] using quintic spline method.

Table 8.1

Values of frequency parameter Ω for C-C plate for $\mu = -0.5$, $\eta = 1.0$, $\varepsilon = 0.3$

			n = 1						n = 2					
		K^*	0			200			0			200		
Mode	α	p \ G	0	10	25	0	10	25	0	10	25	0	10	25
I	-0.5	0.5	17.7546	22.5574	28.0991	21.6533	25.7417	30.7170	19.4384	23.3583	28.1145	22.6052	26.0544	30.3930
		1	17.9224	22.6938	28.2129	21.7907	25.8611	30.8208	19.6296	23.5224	28.2558	22.7698	26.2015	30.5238
		2	18.2512	22.9628	28.4379	22.0612	26.0968	31.0265	20.0043	23.8452	28.5348	23.0933	26.4915	30.7820
	0	0.5	27.3375	29.5510	32.5622	29.1580	31.2427	34.1046	27.3375	29.5510	32.5622	29.1580	31.2427	34.1046
		1	27.5750	29.7699	32.7598	29.3801	31.4491	34.2928	27.5750	29.7699	32.7598	29.3801	31.4491	34.2928
		2	28.0417	30.2009	33.1498	29.8173	31.8563	34.6645	28.0417	30.2009	33.1498	29.8173	31.8563	34.6645
	0.5	0.5	36.4973	37.7870	39.6337	37.5583	38.8123	40.6118	34.5455	36.0525	38.1864	35.7720	37.2283	39.2970
		1	36.8046	38.0826	39.9138	37.8565	39.0997	40.8847	34.8303	36.3237	38.4402	36.0464	37.4904	39.5431
		2	37.4092	38.6647	40.4662	38.4436	39.6659	41.4233	35.3908	36.8581	38.9412	36.5871	38.0073	40.0293
II	-0.5	0.5	49.6104	56.6494	65.6813	51.1444	57.9979	66.8485	55.1282	60.8021	68.3476	56.3433	61.9062	69.3321
		1	49.8348	56.8498	65.8586	51.3621	58.1937	67.0226	55.3841	61.0382	68.5627	56.5937	62.1382	69.5441
		2	50.2792	57.2472	66.2107	51.7933	58.5819	67.3686	55.8908	61.5063	68.9895	57.0896	62.5980	69.9649
	0	0.5	76.0488	79.1502	83.5714	76.7289	79.8039	84.1907	76.0488	79.1502	83.5714	76.7289	79.8039	84.1907
		1	76.3799	79.4679	83.8717	77.0570	80.1189	84.4888	76.3799	79.4679	83.8717	77.0570	80.1189	84.4888
		2	77.0361	80.0980	84.4678	77.7075	80.7439	85.0805	77.0361	80.0980	84.4678	77.7075	80.7439	85.0805
	0.5	0.5	101.2428	103.0415	105.6771	101.6368	103.4287	106.0546	94.8494	96.9654	100.0465	95.3096	97.4155	100.4828
		1	101.6775	103.4679	106.0918	102.0698	103.8534	106.4678	95.2498	97.3560	100.4236	95.7081	97.8044	100.8583
		2	102.5398	104.3138	106.9147	102.9287	104.6962	107.2878	96.0443	98.1313	101.1724	96.4987	98.5761	101.6037
III	-0.5	0.5	97.8960	105.8752	116.7621	98.6842	106.6045	117.4238	109.4998	115.8949	124.8270	110.1213	116.4822	125.3724
		1	98.1500	106.1130	116.9814	98.9362	106.8406	117.6419	109.7892	116.1714	125.0877	110.4091	116.7573	125.6320
		2	98.6550	106.5861	117.4181	99.4373	107.3105	118.0761	110.3646	116.7213	125.6064	110.9812	117.3044	126.1484
	0	0.5	149.7401	153.1712	158.1746	150.0881	153.5114	158.5041	149.7401	153.1712	158.1746	150.0881	153.5114	158.5041
		1	150.1161	153.5387	158.5303	150.4633	153.8781	158.8590	150.1161	153.5387	158.5303	150.4633	153.8781	158.8590
		2	150.8643	154.2700	159.2383	151.2097	154.6079	159.5656	150.8643	154.2700	159.2383	151.2097	154.6079	159.5656
	0.5	0.5	199.0731	201.0609	204.0047	199.2753	201.2611	204.2019	185.6962	188.0427	191.5051	185.9331	188.2767	191.7349
		1	199.5669	201.5494	204.4855	199.7686	201.7491	204.6823	186.1510	188.4913	191.9447	186.3873	188.7246	192.1739
		2	200.5496	202.5217	205.4427	200.7503	202.7204	205.6386	187.0562	189.3841	192.8198	187.2914	189.6164	193.0480

Table 8.2

Values of frequency parameter Ω for C-C plate for $\mu = 1.0$, $\eta = -0.5$, $\varepsilon = 0.3$

			n = 1						n = 2					
Mode	α	K^*	0			200			0			200		
		p \ G	0	10	25	0	10	25	0	10	25	0	10	25
I	-0.5	0.5	48.0989	53.0770	59.6788	52.2284	56.8398	63.0418	52.7886	56.7326	62.1244	56.0961	59.8176	64.9478
		1	48.5495	53.4875	60.0465	52.6454	57.2249	63.3913	53.2999	57.2109	62.5643	56.5792	60.2729	65.3701
		2	49.4352	54.2962	60.7726	53.4667	57.9845	64.0821	54.3044	58.1522	63.4320	57.5295	61.1699	66.2035
	0	0.5	73.3020	75.5699	78.8368	75.1493	77.3632	80.5575	73.3020	75.5699	78.8368	75.1493	77.3632	80.5575
		1	73.9349	76.1825	79.4220	75.7670	77.9618	81.1305	73.9349	76.1825	79.4220	75.7670	77.9618	81.1305
		2	75.1817	77.3900	80.5769	76.9845	79.1426	82.2618	75.1817	77.3900	80.5769	76.9845	79.1426	82.2618
	0.5	0.5	97.3848	98.7197	100.6846	98.4484	99.7691	101.7138	92.0273	93.6062	95.9188	93.2582	94.8166	97.1004
		1	98.2027	99.5252	101.4725	99.2576	100.5662	102.4938	92.7847	94.3492	96.6415	94.0057	95.5502	97.8144
		2	99.8149	101.1136	103.0271	100.8529	102.1385	104.0331	94.2784	95.8151	98.0685	95.4803	96.9980	99.2244
II	-0.5	0.5	134.5806	141.6231	151.5176	136.1379	143.1036	152.9018	150.0357	155.5814	163.5134	151.2682	156.7701	164.6444
		1	135.1833	142.1979	152.0576	136.7339	143.6726	153.4371	150.7274	156.2506	164.1533	151.9542	157.4343	165.2800
		2	136.3788	143.3387	153.1302	137.9161	144.8019	154.5002	152.0986	157.5781	165.4233	153.3148	158.7521	166.5417
	0	0.5	202.8367	205.9908	210.6268	203.5139	206.6577	211.2791	202.8367	205.9908	210.6268	203.5139	206.6577	211.2791
		1	203.7097	206.8496	211.4658	204.3840	207.5138	212.1155	203.7097	206.8496	211.4658	204.3840	207.5138	212.1155
		2	205.4424	208.5548	213.1319	206.1111	209.2136	213.7766	205.4424	208.5548	213.1319	206.1111	209.2136	213.7766
	0.5	0.5	267.7382	269.6007	272.3685	268.1276	269.9874	272.7513	250.0011	252.2151	255.4960	250.4549	252.6649	255.9401
		1	268.8736	270.7276	273.4828	269.2614	271.1127	273.8640	251.0422	253.2460	256.5124	251.4941	253.6940	256.9546
		2	271.1281	272.9652	275.6958	271.5126	273.3471	276.0739	253.1097	255.2937	258.5313	253.5579	255.7381	258.9701
III	-0.5	0.5	265.6516	273.4885	284.8165	266.4496	274.2637	285.5610	298.2455	304.4078	313.4083	298.8759	305.0255	314.0082
		1	266.3226	274.1421	285.4467	267.1185	274.9155	286.1895	299.0142	305.1628	314.1443	299.6429	305.7789	314.7428
		2	267.6582	275.4434	286.7017	268.4501	276.2130	287.4412	300.5440	306.6656	315.6096	301.1695	307.2787	316.2053
	0	0.5	398.3897	401.8655	407.0220	398.7355	402.2082	407.3604	398.3897	401.8655	407.0220	398.7355	402.2082	407.3604
		1	399.3650	402.8322	407.9762	399.7099	403.1741	408.3138	399.3650	402.8322	407.9762	399.7099	403.1741	408.3138
		2	401.3074	404.7575	409.8767	401.6506	405.0978	410.2127	401.3074	404.7575	409.8767	401.6506	405.0978	410.2127
	0.5	0.5	524.4009	526.4582	529.5285	524.5999	526.6565	529.7257	487.3299	489.7810	493.4335	487.5624	490.0124	493.6632
		1	525.6695	527.7216	530.7841	525.8681	527.9194	530.9808	488.4935	490.9384	494.5817	488.7254	491.1692	494.8108
		2	528.1965	530.2381	533.2853	528.3941	530.4350	533.4810	490.8114	493.2440	496.8691	491.0423	493.4737	497.0971

Table 8.3
Values of frequency parameter Ω for C-C plate for $\mu = 0.0$, $\eta = 0.0$, $\varepsilon = 0.3$

			n = 1						n = 2					
Mode	α	K^*	0			200			0			200		
		p \ G	0	10	25	0	10	25	0	10	25	0	10	25
I	-0.5	0.5	29.4462	35.2259	42.2594	34.1993	39.2767	45.6822	32.2998	36.9643	42.8969	36.1407	40.3559	45.8436
		1	29.7276	35.4661	42.4649	34.4433	39.4935	45.8732	32.6202	37.2497	43.1488	36.4288	40.6187	46.0803
		2	30.2797	35.9394	42.8710	34.9238	39.9213	46.2511	33.2486	37.8112	43.6458	36.9954	41.1366	46.5480
	0	0.5	44.9520	47.5902	51.2679	47.1242	49.6471	53.1827	44.9520	47.5902	51.2679	47.1242	49.6471	53.1827
		1	45.3462	47.9616	51.6114	47.5003	50.0032	53.5139	45.3462	47.9616	51.6114	47.5003	50.0032	53.5139
		2	46.1218	48.6934	52.2895	48.2412	50.7055	54.1682	46.1218	48.6934	52.2895	48.2412	50.7055	54.1682
	0.5	0.5	59.7597	61.2991	63.5303	61.0166	62.5249	64.7137	56.4759	58.2824	60.8788	57.9287	59.6909	62.2282
		1	60.2677	61.7929	64.0050	61.5141	63.0091	65.1797	56.9460	58.7362	61.3108	58.3869	60.1339	62.6507
		2	61.2680	62.7662	64.9413	62.4942	63.9636	66.0992	57.8724	59.6312	62.1638	59.2904	61.0080	63.4854
II	-0.5	0.5	82.2570	90.6167	101.7436	84.0895	92.2827	103.2291	91.5889	98.2714	107.4295	93.0424	99.6271	108.6701
		1	82.6289	90.9585	102.0533	84.4533	92.6184	103.5344	92.0144	98.6724	107.8020	93.4614	100.0227	109.0385
		2	83.3660	91.6367	102.6685	85.1747	93.2847	104.1410	92.8574	99.4675	108.5414	94.2917	100.8074	109.7697
	0	0.5	124.8192	128.4978	133.8127	125.6178	129.2736	134.5579	124.8192	128.4978	133.8127	125.6178	129.2736	134.5579
		1	125.3621	129.0248	134.3182	126.1573	129.7975	135.0607	125.3621	129.0248	134.3182	126.1573	129.7975	135.0607
		2	126.4393	130.0707	135.3221	127.2277	130.8373	136.0590	126.4393	130.0707	135.3221	127.2277	130.8373	136.0590
	0.5	0.5	165.3237	167.4646	170.6215	165.7829	167.9180	171.0665	154.5741	157.1004	160.8082	155.1090	157.6268	161.3224
		1	166.0329	168.1640	171.3069	166.4902	168.6155	171.7501	155.2258	157.7405	161.4320	155.7584	158.2647	161.9442
		2	167.4404	169.5523	172.6676	167.8938	170.0000	173.1073	156.5192	159.0114	162.6710	157.0475	159.5314	163.1793
III	-0.5	0.5	162.2789	171.6943	184.8526	163.2181	172.5823	185.6776	181.8988	189.3988	200.0730	182.6416	190.1124	200.7485
		1	162.6951	172.0909	185.2254	163.6319	172.9769	186.0488	182.3743	189.8588	200.5130	183.1151	190.5706	201.1870
		2	163.5230	172.8804	185.9678	164.4550	173.7623	186.7878	183.3201	190.7741	201.3888	184.0572	191.4825	202.0599
	0	0.5	245.5472	249.6074	255.5737	245.9541	250.0077	255.9647	245.5472	249.6074	255.5737	245.9541	250.0077	255.9647
		1	246.1573	250.2076	256.1599	246.5632	250.6070	256.5500	246.1573	250.2076	256.1599	246.5632	250.6070	256.5500
		2	247.3718	251.4025	257.3271	247.7757	251.8000	257.7155	247.3718	251.4025	257.3271	247.7757	251.8000	257.7155
	0.5	0.5	324.6730	327.0347	330.5445	324.9072	327.2673	330.7745	302.2215	305.0157	309.1573	302.4951	305.2868	309.4247
		1	325.4702	327.8258	331.3267	325.7038	328.0578	331.5562	302.9540	305.7411	309.8722	303.2269	306.0115	310.1390
		2	327.0574	329.4010	332.8843	327.2899	329.6318	333.1128	304.4128	307.1856	311.2960	304.6844	307.4547	311.5616

Table 8.4

Values of frequency parameter Ω for C-C plate for $\mu = -0.5$, $\eta = 1.0$, $\varepsilon = 0.5$

			n = 1						n = 2					
		K^*	0			200			0			200		
Mode	α	p \ G	0	10	25	0	10	25	0	10	25	0	10	25
I	-0.5	0.5	30.8760	36.4249	43.2964	33.2284	38.4398	45.0054	34.0431	38.5971	44.4606	35.9389	40.2791	45.9282
		1	30.9707	36.5053	43.3643	33.3164	38.5160	45.0707	34.1511	38.6930	44.5445	36.0412	40.3710	46.0095
		2	31.1589	36.6654	43.4995	33.4914	38.6677	45.2008	34.3658	38.8837	44.7117	36.2448	40.5539	46.1714
	0	0.5	50.4529	52.6690	55.8121	51.3734	53.5514	56.6455	50.4529	52.6690	55.8121	51.3734	53.5514	56.6455
		1	50.6006	52.8100	55.9445	51.5184	53.6901	56.7759	50.6006	52.8100	55.9445	51.5184	53.6901	56.7759
		2	50.8946	53.0907	56.2081	51.8071	53.9661	57.0356	50.8946	53.0907	56.2081	51.8071	53.9661	57.0356
	0.5	0.5	69.6080	70.7968	72.5394	70.0989	71.2794	73.0104	65.9532	67.3106	69.2910	66.5109	67.8570	69.8217
		1	69.8092	70.9942	72.7316	70.2986	71.4754	73.2013	66.1417	67.4949	69.4693	66.6977	68.0397	69.9986
		2	70.2096	71.3871	73.1141	70.6961	71.8657	73.5813	66.5168	67.8616	69.8242	67.0697	68.4034	70.3508
II	-0.5	0.5	85.6232	93.5479	104.2169	86.5022	94.3531	104.9404	95.2061	101.6576	110.5629	95.9074	102.3148	111.1675
		1	85.7497	93.6640	104.3215	86.6274	94.4682	105.0443	95.3497	101.7927	110.6880	96.0499	102.4490	111.2919
		2	86.0020	93.8956	104.5304	86.8771	94.6979	105.2517	95.6360	102.0622	110.9375	96.3342	102.7168	111.5401
	0	0.5	139.6018	142.6587	147.1163	139.9389	142.9885	147.4362	139.6018	142.6587	147.1163	139.9389	142.9885	147.4362
		1	139.8044	142.8566	147.3079	140.1410	143.1860	147.6274	139.8044	142.8566	147.3079	140.1410	143.1860	147.6274
		2	140.2084	143.2515	147.6902	140.5440	143.5800	148.0089	140.2084	143.2515	147.6902	140.5440	143.5800	148.0089
	0.5	0.5	192.3581	193.9913	196.4136	192.5378	194.1695	196.5897	181.3234	183.1961	185.9664	181.5287	183.3993	186.1666
		1	192.6358	194.2665	196.6851	192.8153	194.4444	196.8609	181.5828	183.4525	186.2187	181.7878	183.6555	186.4186
		2	193.1899	194.8155	197.2268	193.3688	194.9929	197.4020	182.1003	183.9642	186.7220	182.3047	184.1666	186.9214
III	-0.5	0.5	168.3098	177.1925	189.7000	168.7593	177.6195	190.0989	187.8332	195.0437	205.3534	188.1912	195.3884	205.6809
		1	168.4502	177.3263	189.8255	168.8993	177.7529	190.2242	187.9923	195.1975	205.5003	188.3500	195.5420	205.8275
		2	168.7305	177.5935	190.0763	169.1789	178.0195	190.4743	188.3100	195.5047	205.7937	188.6671	195.8486	206.1205
	0	0.5	274.1382	277.5047	282.4773	274.3104	277.6748	282.6444	274.1382	277.5047	282.4773	274.3104	277.6748	282.6444
		1	274.3633	277.7270	282.6955	274.5353	277.8970	282.8625	274.3633	277.7270	282.6955	274.5353	277.8970	282.8625
		2	274.8128	278.1709	283.1315	274.9845	278.3406	283.2982	274.8128	278.1709	283.1315	274.9845	278.3406	283.2982
	0.5	0.5	377.5190	379.3154	381.9936	377.6110	379.4069	382.0845	355.0838	357.1482	360.2216	355.1891	357.2529	360.3254
		1	377.8274	379.6223	382.2983	377.9193	379.7138	382.3891	355.3718	357.4345	360.5053	355.4770	357.5391	360.6090
		2	378.4435	380.2353	382.9067	378.5352	380.3266	382.9974	355.9471	358.0062	361.0719	356.0521	358.1106	361.1754

Table 8.5
Values of frequency parameter Ω for C-C plate for $\mu = 1.0$, $\eta = -0.5$, $\varepsilon = 0.5$

			n = 1						n = 2					
		K^*	0			200			0			200		
Mode	α	p \ G	0	10	25	0	10	25	0	10	25	0	10	25
I	-0.5	0.5	96.1841	101.9368	109.9402	98.5920	104.2106	112.0501	106.2975	110.9032	117.4329	108.2259	112.7516	119.1783
		1	96.4768	102.2125	110.1952	98.8778	104.4804	112.3004	106.6310	111.2229	117.7347	108.5537	113.0662	119.4759
		2	97.0593	102.7613	110.7031	99.4465	105.0177	112.7990	107.2945	111.8589	118.3356	109.2059	113.6923	120.0683
	0	0.5	156.2404	158.4918	161.8042	157.1646	159.4030	162.6968	156.2404	158.4918	161.8042	157.1646	159.4030	162.6968
		1	156.6965	158.9408	162.2431	157.6180	159.8494	163.1333	156.6965	158.9408	162.2431	157.6180	159.8494	163.1333
		2	157.6041	159.8344	163.1169	158.5203	160.7380	164.0024	157.6041	159.8344	163.1169	158.5203	160.7380	164.0024
	0.5	0.5	214.9960	216.2070	218.0095	215.4868	216.6950	218.4935	203.4110	204.8058	206.8783	203.9687	205.3597	207.4266
		1	215.6169	216.8240	218.6209	216.1062	217.3107	219.1035	203.9925	205.3830	207.4490	204.5487	205.9353	207.9958
		2	216.8527	218.0522	219.8379	217.3393	218.5361	220.3179	205.1502	206.5319	208.5852	205.7032	207.0812	209.1291
II	-0.5	0.5	266.9443	274.9290	286.4537	267.8294	275.7885	287.2786	297.8208	304.1997	313.5015	298.5279	304.8920	314.1732
		1	267.3373	275.3105	286.8197	268.2211	276.1688	287.6436	298.2691	304.6388	313.9279	298.9751	305.3301	314.5987
		2	268.1213	276.0717	287.5501	269.0026	276.9276	288.3720	299.1634	305.5148	314.7785	299.8673	306.2041	315.4475
	0	0.5	431.1730	434.2604	438.8479	431.5094	434.5944	439.1783	431.1730	434.2604	438.8479	431.5094	434.5944	439.1783
		1	431.7954	434.8781	439.4587	432.1313	435.2116	439.7887	431.7954	434.8781	439.4587	432.1313	435.2116	439.7887
		2	433.0373	436.1106	440.6774	433.3722	436.4431	441.0065	433.0373	436.1106	440.6774	433.3722	436.4431	441.0065
	0.5	0.5	591.6089	593.2713	595.7556	591.7875	593.4495	595.9330	556.2786	558.1998	561.0681	556.4823	558.4028	561.2701
		1	592.4579	594.1178	596.5982	592.6363	594.2957	596.7754	557.0690	558.9873	561.8512	557.2725	559.1901	562.0530
		2	594.1519	595.8067	598.2796	594.3299	595.9841	598.4563	558.6463	560.5586	563.4139	558.8492	560.7609	563.6151
III	-0.5	0.5	524.8828	533.7107	546.6734	525.3353	534.1557	547.1078	588.0437	595.0993	605.5192	588.4052	595.4565	605.8702
		1	525.3153	534.1363	547.0892	525.7675	534.5809	547.5233	588.5360	595.5862	605.9981	588.8972	595.9431	606.3489
		2	526.1792	534.9863	547.9197	526.6306	535.4303	548.3532	589.5193	596.5585	606.9548	589.8798	596.9148	607.3050
	0	0.5	845.7079	849.0991	854.1601	845.8795	849.2701	854.3300	845.7079	849.0991	854.1601	845.8795	849.2701	854.3300
		1	846.3941	849.7826	854.8394	846.5656	849.9534	855.0092	846.3941	849.7826	854.8394	846.5656	849.9534	855.0092
		2	847.7649	851.1478	856.1963	847.9361	851.3183	856.3659	847.7649	851.1478	856.1963	847.9361	851.3183	856.3659
	0.5	0.5	1158.9604	1160.7882	1163.5242	1159.0517	1160.8793	1163.6151	1086.9667	1089.0827	1092.2488	1087.0709	1089.1867	1092.3525
		1	1159.8965	1161.7227	1164.4565	1159.9877	1161.8137	1164.5473	1087.8383	1089.9526	1093.1160	1087.9424	1090.0565	1093.2196
		2	1161.7662	1163.5894	1166.3186	1161.8572	1163.6803	1166.4093	1089.5794	1091.6901	1094.8483	1089.6833	1091.7939	1094.9517

Table 8.6

Values of frequency parameter Ω for C-C plate for $\mu = 0.0$, $\eta = 0.0$, $\varepsilon = 0.5$

			n = 1						n = 2					
Mode	α	K^*	0			200			0			200		
		p \ G	0	10	25	0	10	25	0	10	25	0	10	25
I	-0.5	0.5	54.7228	61.5556	70.4323	57.6077	64.1316	72.6919	60.4458	66.0004	73.4489	62.7660	68.1291	75.3646
		1	54.8912	61.7053	70.5632	57.7679	64.2755	72.8189	60.6381	66.1770	73.6083	62.9513	68.3004	75.5200
		2	55.2262	62.0034	70.8241	58.0866	64.5620	73.0719	61.0205	66.5285	73.9258	63.3201	68.6413	75.8298
	0	0.5	88.9892	91.6847	95.5714	90.1059	92.7690	96.6121	88.9892	91.6847	95.5714	90.1059	92.7690	96.6121
		1	89.2508	91.9380	95.8135	90.3642	93.0193	96.8516	89.2508	91.9380	95.8135	90.3642	93.0193	96.8516
		2	89.7713	92.4421	96.2956	90.8784	93.5176	97.3286	89.7713	92.4421	96.2956	90.8784	93.5176	97.3286
	0.5	0.5	122.5098	123.9551	126.0886	123.1036	124.5419	126.6655	115.9318	117.5871	120.0222	116.6062	118.2520	120.6736
		1	122.8654	124.3060	126.4330	123.4574	124.8912	127.0084	116.2646	117.9147	120.3424	116.9370	118.5777	120.9920
		2	123.5730	125.0046	127.1186	124.1616	125.5865	127.6909	116.9269	118.5667	120.9798	117.5955	119.2260	121.6260
II	-0.5	0.5	151.7207	161.3655	174.7466	152.7893	162.3705	175.6748	169.0444	176.8448	187.8746	169.8988	177.6617	188.6435
		1	151.9449	161.5765	174.9418	153.0119	162.5802	175.8690	169.2995	177.0893	188.1056	170.1527	177.9051	188.8736
		2	152.3920	161.9975	175.3315	153.4559	162.9986	176.2567	169.8083	177.5770	188.5665	170.6590	178.3906	189.3327
	0	0.5	245.9860	249.6911	255.1407	246.3922	250.0913	255.5324	245.9860	249.6911	255.1407	246.3922	250.0913	255.5324
		1	246.3428	250.0423	255.4839	246.7484	250.4419	255.8751	246.3428	250.0423	255.4839	246.7484	250.4419	255.8751
		2	247.0545	250.7429	256.1688	247.4589	251.1414	256.5589	247.0545	250.7429	256.1688	247.4589	251.1414	256.5589
	0.5	0.5	338.0902	340.0720	343.0217	338.3060	340.2866	343.2343	318.2184	320.4959	323.8795	318.4645	320.7402	324.1212
		1	338.5780	340.5567	343.5018	338.7934	340.7709	343.7142	318.6732	320.9471	324.3255	318.9188	321.1911	324.5669
		2	339.5511	341.5237	344.4599	339.7659	341.7373	344.6717	319.5805	321.8474	325.2157	319.8255	322.0907	325.4564
III	-0.5	0.5	298.2008	308.9539	324.3790	298.7468	309.4810	324.8810	333.5068	342.1949	354.7976	333.9432	342.6202	355.2079
		1	298.4481	309.1931	324.6074	298.9937	309.7198	325.1091	333.7878	342.4693	355.0633	334.2238	342.8943	355.4732
		2	298.9421	309.6709	325.0638	299.4867	310.1967	325.5647	334.3488	343.0175	355.5938	334.7841	343.4418	356.0031
	0	0.5	482.8294	486.9037	492.9505	483.0364	487.1090	493.1533	482.8294	486.9037	492.9505	483.0364	487.1090	493.1533
		1	483.2237	487.2947	493.3366	483.4306	487.4999	493.5393	483.2237	487.2947	493.3366	483.4306	487.4999	493.5393
		2	484.0113	488.0757	494.1079	484.2179	488.2805	494.3102	484.0113	488.0757	494.1079	484.2179	488.2805	494.3102
	0.5	0.5	663.1353	665.3130	668.5658	663.2453	665.4227	668.6750	622.7426	625.2497	628.9907	622.8683	625.3749	629.1152
		1	663.6744	665.8502	669.1004	663.7844	665.9599	669.2095	623.2453	625.7502	629.4882	623.3709	625.8753	629.6125
		2	664.7511	666.9234	670.1681	664.8610	667.0328	670.2770	624.2494	626.7501	630.4818	624.3748	626.8750	630.6059

Table 8.7

Values of frequency parameter Ω for C-S plate for $\mu = -0.5$, $\eta = 1.0$, $\varepsilon = 0.3$

			n = 1						n = 2					
		K^*	0			200			0			200		
Mode	α	p \ G	0	10	25	0	10	25	0	10	25	0	10	25
I	-0.5	0.5	12.5029	18.7694	25.0645	17.5663	22.4726	27.9530	14.1308	19.3588	24.9303	18.2469	22.5402	27.4749
		1	12.6709	18.8940	25.1676	17.6858	22.5765	28.0453	14.3202	19.5102	25.0591	18.3939	22.6703	27.5918
		2	12.9988	19.1394	25.3715	17.9214	22.7816	28.2279	14.6897	19.8077	25.3132	18.6830	22.9268	27.8227
	0	0.5	17.7661	20.7793	24.5753	20.3338	23.0189	26.5022	17.7661	20.7793	24.5753	20.3338	23.0189	26.5022
		1	18.0210	21.0001	24.7652	20.5561	23.2178	26.6778	18.0210	21.0001	24.7652	20.5561	23.2178	26.6778
		2	18.5190	21.4338	25.1396	20.9926	23.6095	27.0246	18.5190	21.4338	25.1396	20.9926	23.6095	27.0246
	0.5	0.5	22.7116	24.4714	26.8884	24.2660	25.9219	28.2169	20.8496	22.8860	25.6301	22.6658	24.5538	27.1319
		1	23.0582	24.7938	27.1829	24.5902	26.2261	28.4972	21.1826	23.1900	25.9022	22.9719	24.8369	27.3887
		2	23.7350	25.4255	27.7616	25.2250	26.8232	29.0489	21.8325	23.7856	26.4374	23.5714	25.3929	27.8945
II	-0.5	0.5	40.4809	48.8035	58.9419	42.3430	50.3583	60.2360	45.5360	52.4020	61.1028	47.0037	53.6826	62.2049
		1	40.7035	48.9952	59.1085	42.5557	50.5440	60.3991	45.7879	52.6283	61.3057	47.2478	53.9035	62.4041
		2	41.1435	49.3752	59.4394	42.9767	50.9124	60.7228	46.2859	53.0765	61.7080	47.7305	54.3412	62.7993
	0	0.5	60.6804	64.2257	69.2002	61.5213	65.0197	69.9369	60.6804	64.2257	69.2002	61.5213	65.0197	69.9369
		1	61.0156	64.5431	69.4958	61.8519	65.3332	70.2293	61.0156	64.5431	69.4958	61.8519	65.3332	70.2293
		2	61.6791	65.1719	70.0819	62.5063	65.9544	70.8092	61.6791	65.1719	70.0819	62.5063	65.9544	70.8092
	0.5	0.5	79.9644	81.9755	84.9020	80.4559	82.4546	85.3642	74.3272	76.6620	80.0339	74.9053	77.2221	80.5696
		1	80.4101	82.4101	85.3213	80.8988	82.8866	85.7811	74.7417	77.0635	80.4178	75.3165	77.6205	80.9509
		2	81.2927	83.2709	86.1523	81.7759	83.7424	86.6076	75.5626	77.8589	81.1790	76.1310	78.4101	81.7069
III	-0.5	0.5	84.7248	93.7976	105.8659	85.6336	94.6189	106.5940	95.4034	102.8256	112.9578	96.1176	103.4886	113.5616
		1	84.9784	94.0308	106.0778	85.8845	94.8500	106.8045	95.6908	103.0965	113.2102	96.4029	103.7578	113.8126
		2	85.4821	94.4944	106.4996	86.3829	95.3097	107.2234	96.2618	103.6351	113.7121	96.9697	104.2929	114.3119
	0	0.5	128.2152	131.9810	137.4352	128.6195	132.3736	137.8120	128.2152	131.9810	137.4352	128.6195	132.3736	137.8120
		1	128.5956	132.3508	137.7905	128.9987	132.7422	138.1663	128.5956	132.3508	137.7905	128.9987	132.7422	138.1663
		2	129.3520	133.0860	138.4973	129.7527	133.4753	138.8712	129.3520	133.0860	138.4973	129.7527	133.4753	138.8712
	0.5	0.5	169.6623	171.8062	174.9717	169.8981	172.0389	175.2002	157.6431	160.1506	163.8374	157.9205	160.4236	164.1041
		1	170.1657	172.3029	175.4590	170.4008	172.5350	175.6868	158.1092	160.6090	164.2846	158.3858	160.8812	164.5506
		2	171.1667	173.2910	176.4284	171.4004	173.5217	176.6550	159.0363	161.5207	165.1745	159.3113	161.7913	165.4390

Table 8.8

Values of frequency parameter Ω for C-S plate for $\mu = 1.0$, $\eta = -0.5$, $\varepsilon = 0.3$

			n = 1						n = 2					
		K^*	0			200			0			200		
Mode	α	p \ G	0	10	25	0	10	25	0	10	25	0	10	25
I	-0.5	0.5	33.9253	40.8742	49.2245	39.8777	45.8987	53.4347	38.5134	44.0621	51.0663	43.2017	48.1875	54.6373
		1	34.4067	41.2840	49.5760	40.2900	46.2659	53.7602	39.0466	44.5383	51.4889	43.6798	48.6252	55.0341
		2	35.3479	42.0897	50.2697	41.1007	46.9899	54.4037	40.0891	45.4731	52.3213	44.6182	49.4865	55.8169
	0	0.5	47.3170	50.5069	54.9231	50.2062	53.2219	57.4279	47.3170	50.5069	54.9231	50.2062	53.2219	57.4279
		1	48.0689	51.2125	55.5735	50.9156	53.8921	58.0503	48.0689	51.2125	55.5735	50.9156	53.8921	58.0503
		2	49.5375	52.5944	56.8509	52.3047	55.2072	59.2747	49.5375	52.5944	56.8509	52.3047	55.2072	59.2747
	0.5	0.5	59.8682	61.7269	64.4107	61.5956	63.4036	66.0192	54.7514	56.9385	60.0652	56.7908	58.9023	61.9300
		1	60.9152	62.7425	65.3841	62.6137	64.3927	66.9692	55.7757	57.9235	60.9990	57.7790	59.8550	62.8361
		2	62.9562	64.7251	67.2879	64.6011	66.3261	68.8293	57.7694	59.8445	62.8244	59.7058	61.7159	64.6096
II	-0.5	0.5	109.9679	118.4759	130.1160	111.9116	120.2856	131.7679	124.3051	131.1261	140.6692	125.8348	132.5789	142.0255
		1	110.5832	119.0521	130.6473	112.5165	120.8534	132.2927	125.0034	131.7933	141.2983	126.5248	133.2390	142.6487
		2	111.8019	120.1947	131.7021	113.7147	121.9793	133.3347	126.3861	133.1155	142.5459	127.8913	134.5472	143.8848
	0	0.5	161.5124	165.1385	170.4321	162.3683	165.9760	171.2440	161.5124	165.1385	170.4321	162.3683	165.9760	171.2440
		1	162.4348	166.0410	171.3070	163.2859	166.8740	172.1149	162.4348	166.0410	171.3070	163.2859	166.8740	172.1149
		2	164.2625	167.8300	173.0423	165.1044	168.6542	173.8422	164.2625	167.8300	173.0423	165.1044	168.6542	173.8422
	0.5	0.5	210.6459	212.7322	215.8235	211.1415	213.2230	216.3073	194.8187	197.2696	200.8887	195.4000	197.8437	201.4525
		1	211.8704	213.9446	217.0183	212.3632	214.4326	217.4994	195.9585	198.3950	201.9935	196.5364	198.9658	202.5542
		2	214.2973	216.3478	219.3873	214.7845	216.8304	219.8632	198.2175	200.6260	204.1843	198.7888	201.1905	204.7390
III	-0.5	0.5	230.1292	239.1377	252.0128	231.0613	240.0361	252.8667	260.4136	267.6249	278.0593	261.1486	268.3410	278.7494
		1	230.8120	239.7976	252.6429	231.7413	240.6935	253.4947	261.1899	268.3832	278.7932	261.9228	269.0972	279.4815
		2	232.1702	241.1107	253.8973	233.0940	242.0016	254.7448	262.7340	269.8918	280.2539	263.4625	270.6018	280.9386
	0	0.5	340.7512	344.5748	350.2313	341.1569	344.9759	350.6261	340.7512	344.5748	350.2313	341.1569	344.9759	350.6261
		1	341.7653	345.5775	351.2180	342.1697	345.9775	351.6116	341.7653	345.5775	351.2180	342.1697	345.9775	351.6116
		2	343.7831	347.5733	353.1818	344.1851	347.9710	353.5733	343.7831	347.5733	353.1818	344.1851	347.9710	353.5733
	0.5	0.5	445.9881	448.2078	451.5164	446.2223	448.4409	451.7478	412.4245	415.0472	418.9494	412.6990	415.3201	419.2197
		1	447.3232	449.5361	452.8346	447.5568	449.7685	453.0654	413.6597	416.2743	420.1646	413.9334	416.5463	420.4341
		2	449.9805	452.1799	455.4585	450.2127	452.4109	455.6879	416.1182	418.7168	422.5835	416.3903	418.9872	422.8514

Table 8.9

Values of frequency parameter Ω for C-S plate for $\mu = 0.0$, $\eta = 0.0$, $\varepsilon = 0.3$

			n = 1						n = 2					
Mode	α	K^*	0			200			0			200		
		p \ G	0	10	25	0	10	25	0	10	25	0	10	25
I	-0.5	0.5	20.9796	28.7593	37.0568	27.4444	33.7254	40.9994	23.7489	30.1383	37.3646	28.9422	34.3445	40.8031
		1	21.2689	28.9856	37.2459	27.6679	33.9200	41.1715	24.0735	30.4093	37.5981	29.2110	34.5842	41.0183
		2	21.8341	29.4312	37.6196	28.1081	34.3043	41.5120	24.7073	30.9419	38.0585	29.7393	35.0566	41.4432
	0	0.5	29.5386	33.2047	37.9980	32.7495	36.0909	40.5444	29.5386	33.2047	37.9980	32.7495	36.0909	40.5444
		1	29.9777	33.5983	38.3454	33.1461	36.4533	40.8702	29.9777	33.5983	38.3454	33.1461	36.4533	40.8702
		2	30.8359	34.3708	39.0298	33.9243	37.1665	41.5130	30.8359	34.3708	39.0298	33.9243	37.1665	41.5130
	0.5	0.5	37.5741	39.7084	42.7015	39.5037	41.5394	44.4099	34.4397	36.9256	40.3580	36.7030	39.0458	42.3075
		1	38.1733	40.2761	43.2304	40.0739	42.0823	44.9186	35.0181	37.4656	40.8527	37.2462	39.5568	42.7796
		2	39.3433	41.3873	44.2683	41.1898	43.1468	45.9182	36.1465	38.5225	41.8242	38.3088	40.5590	43.7080
II	-0.5	0.5	67.3823	77.4433	90.2402	69.6358	79.4149	91.9373	75.9911	84.2075	95.0218	77.7709	85.8190	96.4522
		1	67.7557	77.7764	90.5359	69.9973	79.7398	92.2276	76.4146	84.5981	95.3784	78.1848	86.2024	96.8036
		2	68.4948	78.4369	91.1229	70.7131	80.3843	92.8041	77.2526	85.3719	96.0858	79.0042	86.9621	97.5009
	0	0.5	99.8632	104.1175	110.1874	100.8596	105.0735	111.0912	99.8632	104.1175	110.1874	100.8596	105.0735	111.0912
		1	100.4228	104.6552	110.6967	101.4137	105.6064	111.5965	100.4228	104.6552	110.6967	101.4137	105.6064	111.5965
		2	101.5312	105.7208	111.7071	102.5114	106.6625	112.5987	101.5312	105.7208	111.7071	102.5114	106.6625	112.5987
	0.5	0.5	130.8413	133.2583	136.8031	131.4185	133.8250	137.3550	121.2953	124.1108	128.2163	121.9719	124.7718	128.8561
		1	131.5838	133.9872	137.5130	132.1577	134.5508	138.0621	121.9854	124.7850	128.8686	122.6582	125.4425	129.5050
		2	133.0546	135.4317	138.9205	133.6222	135.9892	139.4640	123.3528	126.1213	130.1621	124.0180	126.7718	130.7922
III	-0.5	0.5	140.7255	151.5646	166.3975	141.8149	152.5781	167.3222	158.8747	167.6828	179.9903	159.7346	168.4989	180.7517
		1	141.1443	151.9582	166.7623	142.2305	152.9691	167.6851	159.3503	168.1383	180.4212	160.2076	168.9522	181.1807
		2	141.9770	152.7412	167.4886	143.0568	153.7469	168.4074	160.2959	169.0441	181.2786	161.1481	169.8535	182.0345
	0	0.5	210.5050	214.9968	221.5628	210.9795	215.4614	222.0136	210.5050	214.9968	221.5628	210.9795	215.4614	222.0136
		1	211.1294	215.6084	222.1567	211.6025	216.0717	222.6064	211.1294	215.6084	222.1567	211.6025	216.0717	222.6064
		2	212.3713	216.8253	223.3388	212.8417	217.2860	223.7861	212.3713	216.8253	223.3388	212.8417	217.2860	223.7861
	0.5	0.5	276.9330	279.4967	283.2981	277.2070	279.7682	283.5659	256.6706	259.6757	264.1176	256.9917	259.9931	264.4296
		1	277.7565	280.3124	284.1025	278.0297	280.5831	284.3696	257.4324	260.4283	264.8570	257.7526	260.7448	265.1681
		2	279.3949	281.9355	285.7034	279.6665	282.2046	285.9690	258.9483	261.9261	266.3287	259.2666	262.2407	266.6381

Table 8.10
Values of frequency parameter Ω for C-S plate for $\mu = -0.5$, $\eta = 1.0$, $\varepsilon = 0.5$

			n = 1						n = 2					
		K^*	0			200			0			200		
Mode	α	p \ G	0	10	25	0	10	25	0	10	25	0	10	25
I	-0.5	0.5	21.7157	29.0695	37.1735	24.9317	31.5471	39.1436	24.6579	30.8296	37.9771	27.2304	32.9215	39.6925
		1	21.8180	29.1483	37.2374	25.0209	31.6197	39.2043	24.7715	30.9235	38.0562	27.3334	33.0094	39.7682
		2	22.0211	29.3051	37.3647	25.1981	31.7643	39.3252	24.9972	31.1101	38.2139	27.5381	33.1843	39.9192
	0	0.5	33.4584	36.4850	40.5775	34.7843	37.7059	41.6802	33.4584	36.4850	40.5775	34.7843	37.7059	41.6802
		1	33.6303	36.6429	40.7197	34.9496	37.8586	41.8186	33.6303	36.6429	40.7197	34.9496	37.8586	41.8186
		2	33.9715	36.9565	41.0026	35.2780	38.1622	42.0940	33.9715	36.9565	41.0026	35.2780	38.1622	42.0940
	0.5	0.5	44.8905	46.5119	48.8389	45.6111	47.2080	49.5027	41.5658	43.3998	46.0094	42.3933	44.1934	46.7592
		1	45.1345	46.7474	49.0631	45.8512	47.4400	49.7239	41.8027	43.6266	46.2232	42.6257	44.4162	46.9696
		2	45.6184	47.2146	49.5083	46.3276	47.9005	50.1632	42.2725	44.0767	46.6478	43.0865	44.8583	47.3875
II	-0.5	0.5	69.8268	79.1666	91.2876	70.9005	80.1151	92.1113	78.4672	86.2474	96.6358	79.3198	87.0240	97.3296
		1	69.9567	79.2824	91.3895	71.0284	80.2295	92.2124	78.6130	86.3816	96.7575	79.4639	87.1570	97.4505
		2	70.2155	79.5134	91.5930	71.2834	80.4578	92.4140	78.9035	86.6492	97.0005	79.7514	87.4222	97.6917
	0	0.5	111.9461	115.4498	120.5124	112.3626	115.8535	120.8990	111.9461	115.4498	120.5124	112.3626	115.8535	120.8990
		1	112.1592	115.6564	120.7105	112.5749	116.0595	121.0964	112.1592	115.6564	120.7105	112.5749	116.0595	121.0964
		2	112.5839	116.0685	121.1055	112.9981	116.4701	121.4902	112.5839	116.0685	121.1055	112.9981	116.4701	121.4902
	0.5	0.5	153.1615	155.0041	157.7274	153.3847	155.2246	157.9440	143.4980	145.5855	148.6611	143.7542	145.8380	148.9082
		1	153.4572	155.2963	158.0145	153.6800	155.5164	158.2307	143.7771	145.8605	148.9303	144.0328	146.1125	149.1770
		2	154.0468	155.8788	158.5869	154.2687	156.0980	158.8023	144.3335	146.4089	149.4671	144.5883	146.6599	149.7129
III	-0.5	0.5	145.6110	155.6507	169.5512	146.1300	156.1363	169.9970	163.4123	171.7219	183.4290	163.8246	172.1144	183.7965
		1	145.7544	155.7856	169.6761	146.2729	156.2708	170.1215	163.5737	171.8764	183.5749	163.9856	172.2685	183.9421
		2	146.0408	156.0550	169.9254	146.5583	156.5393	170.3702	163.8959	172.1848	183.8662	164.3070	172.5763	184.2329
	0	0.5	235.2544	238.9555	244.4016	235.4543	239.1523	244.5939	235.2544	238.9555	244.4016	235.4543	239.1523	244.5939
		1	235.4879	239.1854	244.6264	235.6875	239.3820	244.8185	235.4879	239.1854	244.6264	235.6875	239.3820	244.8185
		2	235.9540	239.6444	245.0752	236.1533	239.8406	245.2670	235.9540	239.6444	245.0752	236.1533	239.8406	245.2670
	0.5	0.5	322.9018	324.8523	327.7559	323.0088	324.9586	327.8613	302.8091	305.0311	308.3334	302.9319	305.1530	308.4539
		1	323.2242	325.1726	328.0733	323.3311	325.2789	328.1786	303.1119	305.3316	308.6306	303.2346	305.4534	308.7511
		2	323.8678	325.8123	328.7072	323.9745	325.9183	328.8123	303.7167	305.9318	309.2241	303.8391	306.0533	309.3443

Table 8.11

Values of frequency parameter Ω for C-S plate for $\mu = 1.0$, $\eta = -0.5$, $\varepsilon = 0.5$

			n = 1						n = 2					
		K^*	0			200			0			200		
Mode	α	p \ G	0	10	25	0	10	25	0	10	25	0	10	25
I	-0.5	0.5	67.7484	75.7826	86.2575	71.2740	78.9409	89.0344	77.2195	83.7267	92.4862	79.9901	86.2809	94.7953
		1	68.0839	76.0833	86.5230	71.5931	79.2298	89.2918	77.5879	84.0679	92.7972	80.3460	86.6122	95.0989
		2	68.7498	76.6812	87.0514	72.2268	79.8042	89.8041	78.3191	84.7460	93.4158	81.0527	87.2709	95.7030
	0	0.5	103.3284	106.4606	110.9823	104.7466	107.8373	112.3033	103.3284	106.4606	110.9823	104.7466	107.8373	112.3033
		1	103.9049	107.0198	111.5182	105.3154	108.3894	112.8329	103.9049	107.0198	111.5182	105.3154	108.3894	112.8329
		2	105.0483	108.1293	112.5821	106.4436	109.4850	113.8846	105.0483	108.1293	112.5821	106.4436	109.4850	113.8846
	0.5	0.5	137.9446	139.6128	142.0767	138.7124	140.3714	142.8223	127.3385	129.2459	132.0536	128.2242	130.1187	132.9079
		1	138.7734	140.4314	142.8808	139.5366	141.1857	143.6222	128.1524	130.0475	132.8376	129.0325	130.9149	133.6869
		2	140.4159	142.0541	144.4750	141.1703	142.7998	145.2083	129.7643	131.6354	134.3915	130.6336	132.4925	135.2310
II	-0.5	0.5	217.8814	227.4278	240.9996	218.9822	228.4836	241.9970	245.9214	253.7042	264.9114	246.7966	254.5533	265.7253
		1	218.2967	227.8265	241.3768	219.3955	228.8804	242.3727	246.3880	254.1575	265.3469	247.2616	255.0051	266.1595
		2	219.1249	228.6216	242.1293	220.2196	229.6719	243.1222	247.3184	255.0615	266.2156	248.1887	255.9062	267.0256
	0	0.5	345.4159	348.9662	354.2244	345.8378	349.3839	354.6359	345.4159	348.9662	354.2244	345.8378	349.3839	354.6359
		1	346.0964	349.6398	354.8879	346.5175	350.0566	355.2987	346.0964	349.6398	354.8879	346.5175	350.0566	355.2987
		2	347.4533	350.9829	356.2113	347.8727	351.3982	356.6205	347.4533	350.9829	356.2113	347.8727	351.3982	356.6205
	0.5	0.5	470.1900	472.0675	474.8699	470.4151	472.2918	475.0928	439.0599	441.2046	444.4020	439.3178	441.4612	444.6568
		1	471.1341	473.0079	475.8046	471.3588	473.2317	476.0271	439.9515	442.0919	445.2828	440.2089	442.3480	445.5371
		2	473.0166	474.8829	477.6686	473.2404	475.1058	477.8902	441.7294	443.8610	447.0392	441.9857	444.1161	447.2925
III	-0.5	0.5	454.3342	464.3807	479.0409	454.8613	464.8966	479.5415	512.2235	520.4025	532.4192	512.6439	520.8164	532.8241
		1	454.7849	464.8220	479.4694	455.3114	465.3375	479.9695	512.7314	520.9030	532.9093	513.1514	521.3166	533.3138
		2	455.6847	465.7034	480.3251	456.2102	466.2178	480.8243	513.7456	521.9025	533.8880	514.1647	522.3153	534.2918
	0	0.5	725.3570	729.0902	734.6544	725.5576	729.2898	734.8525	725.3570	729.0902	734.6544	725.5576	729.2898	734.8525
		1	726.0872	729.8167	735.3753	726.2876	730.0161	735.5732	726.0872	729.8167	735.3753	726.2876	730.0161	735.5732
		2	727.5452	731.2672	736.8148	727.7452	731.4662	737.0124	727.5452	731.2672	736.8148	727.7452	731.4662	737.0124
	0.5	0.5	990.2750	992.2599	995.2297	990.3819	992.3665	995.3360	925.5469	927.8254	931.2325	925.6693	927.9475	931.3541
		1	991.2811	993.2639	996.2306	991.3879	993.3705	996.3369	926.4917	928.7678	932.1714	926.6139	928.8897	932.2928
		2	993.2901	995.2689	998.2295	993.3967	995.3752	998.3356	928.3783	930.6497	934.0462	928.5002	930.7713	934.1674

Table 8.12

Values of frequency parameter Ω for C-S plate for $\mu = 0.0$, $\eta = 0.0$, $\varepsilon = 0.5$

			n = 1						n = 2					
Mode	α	K^*	0			200			0			200		
		p \ G	0	10	25	0	10	25	0	10	25	0	10	25
I	-0.5	0.5	38.8193	48.1121	59.0127	42.8925	51.4419	61.7458	44.1529	51.8483	61.2942	47.3886	54.6174	63.6409
		1	39.0056	48.2647	59.1399	43.0613	51.5849	61.8675	44.3594	52.0272	61.4491	47.5813	54.7874	63.7902
		2	39.3754	48.5685	59.3932	43.3968	51.8694	62.1099	44.7693	52.3828	61.7574	47.9641	55.1256	64.0876
	0	0.5	59.5060	63.2369	68.4252	61.1634	64.7990	69.8714	59.5060	63.2369	68.4252	61.1634	64.7990	69.8714
		1	59.8199	63.5323	68.6982	61.4689	65.0873	70.1387	59.8199	63.5323	68.6982	61.4689	65.0873	70.1387
		2	60.4429	64.1189	69.2408	62.0753	65.6600	70.6703	60.4429	64.1189	69.2408	62.0753	65.6600	70.6703
	0.5	0.5	79.6433	81.6339	84.5275	80.5414	82.5105	85.3745	73.6438	75.9045	79.1699	74.6768	76.9073	80.1320
		1	80.0898	82.0694	84.9479	80.9831	82.9414	85.7907	74.0786	76.3262	79.5738	75.1057	77.3235	80.5311
		2	80.9755	82.9334	85.7823	81.8590	83.7964	86.6171	74.9406	77.1624	80.3753	75.9560	78.1491	81.3232
II	-0.5	0.5	124.0725	135.5853	151.0975	125.3891	136.7923	152.1821	139.7591	149.2776	162.3961	140.8074	150.2604	163.3005
		1	124.3055	135.7999	151.2918	125.6196	137.0050	152.3750	140.0209	149.5244	162.6253	141.0673	150.5056	163.5284
		2	124.7700	136.2279	151.6796	126.0794	137.4293	152.7601	140.5429	150.0167	163.0826	141.5854	150.9947	163.9831
	0	0.5	197.6720	201.9528	208.2069	198.1773	202.4474	208.6867	197.6720	201.9528	208.2069	198.1773	202.4474	208.6867
		1	198.0535	202.3262	208.5693	198.5577	202.8199	209.0482	198.0535	202.3262	208.5693	198.5577	202.8199	209.0482
		2	198.8140	203.0709	209.2919	199.3164	203.5628	209.7692	198.8140	203.0709	209.2919	199.3164	203.5628	209.7692
	0.5	0.5	269.6814	271.9327	275.2748	269.9510	272.2000	275.5389	252.2083	254.7644	258.5507	252.5170	255.0700	258.8518
		1	270.2104	272.4572	275.7930	270.4794	272.7240	276.0566	252.7073	255.2584	259.0374	253.0154	255.5634	259.3379
		2	271.2650	273.5031	276.8262	271.5330	273.7689	277.0888	253.7025	256.2435	260.0080	254.0094	256.5473	260.3074
III	-0.5	0.5	258.3402	270.5977	287.9572	258.9733	271.2026	288.5261	290.6287	300.7255	315.2118	291.1337	301.2140	315.6783
		1	258.5950	270.8418	288.1877	259.2275	271.4462	288.7562	290.9158	301.0040	315.4789	291.4203	301.4921	315.9450
		2	259.1039	271.3294	288.6482	259.7351	271.9326	289.2158	291.4890	301.5600	316.0122	291.9926	302.0472	316.4776
	0	0.5	414.7431	419.2474	425.9140	414.9841	419.4858	426.1487	414.7431	419.2474	425.9140	414.9841	419.4858	426.1487
		1	415.1567	419.6566	426.3168	415.3975	419.8948	426.5513	415.1567	419.6566	426.3168	415.3975	419.8948	426.5513
		2	415.9825	420.4736	427.1212	416.2229	420.7114	427.3552	415.9825	420.4736	427.1212	416.2229	420.7114	427.3552
	0.5	0.5	567.6355	570.0112	573.5561	567.7639	570.1391	573.6831	531.3697	534.0805	538.1205	531.5167	534.2266	538.2656
		1	568.2058	570.5790	574.1203	568.3341	570.7068	574.2473	531.9052	534.6131	538.6491	532.0520	534.7592	538.7940
		2	569.3444	571.7129	575.2471	569.4725	571.8404	575.3738	532.9745	535.6769	539.7047	533.1210	535.8226	539.8493

Table 8.13

Comparison of frequency parameter Ω for isotropic/polar orthotropic homogeneous ($\mu = 0.0$, $\eta = 0.0$) annular plates of uniform thickness ($\alpha = 0.0$) for $\varepsilon = 0.3$, $\upsilon_\theta = 0.3$

Edge Conds.	Mode	p = 1				p = 2			
		K = 0.01		K = 0.02		K = 0.01		K = 0.02	
C-C	I	46.5347	46.5259*	47.6936	47.6848*	47.3478	47.3359*	48.5430	48.5313*
	II	125.7969	126.0531*	126.2302	126.4855*	126.8917	127.1389*	127.3424	127.5888*
	III	246.3790	246.9155*	246.5995	247.1366*	247.6033	248.1332*	247.8336	248.3640*
C-S	I	31.7469	31.7385*	33.4225	33.4146*	32.6413	32.6310*	34.3519	34.3422*
	II	100.9651	101.1478*	101.5044	101.6862*	102.0940	102.2700*	102.6537	102.8287*
	III	211.3878	211.8208*	211.6457	212.0784*	212.6409	213.0685*	212.9100	213.3373*

* Values taken from Verma[1987].

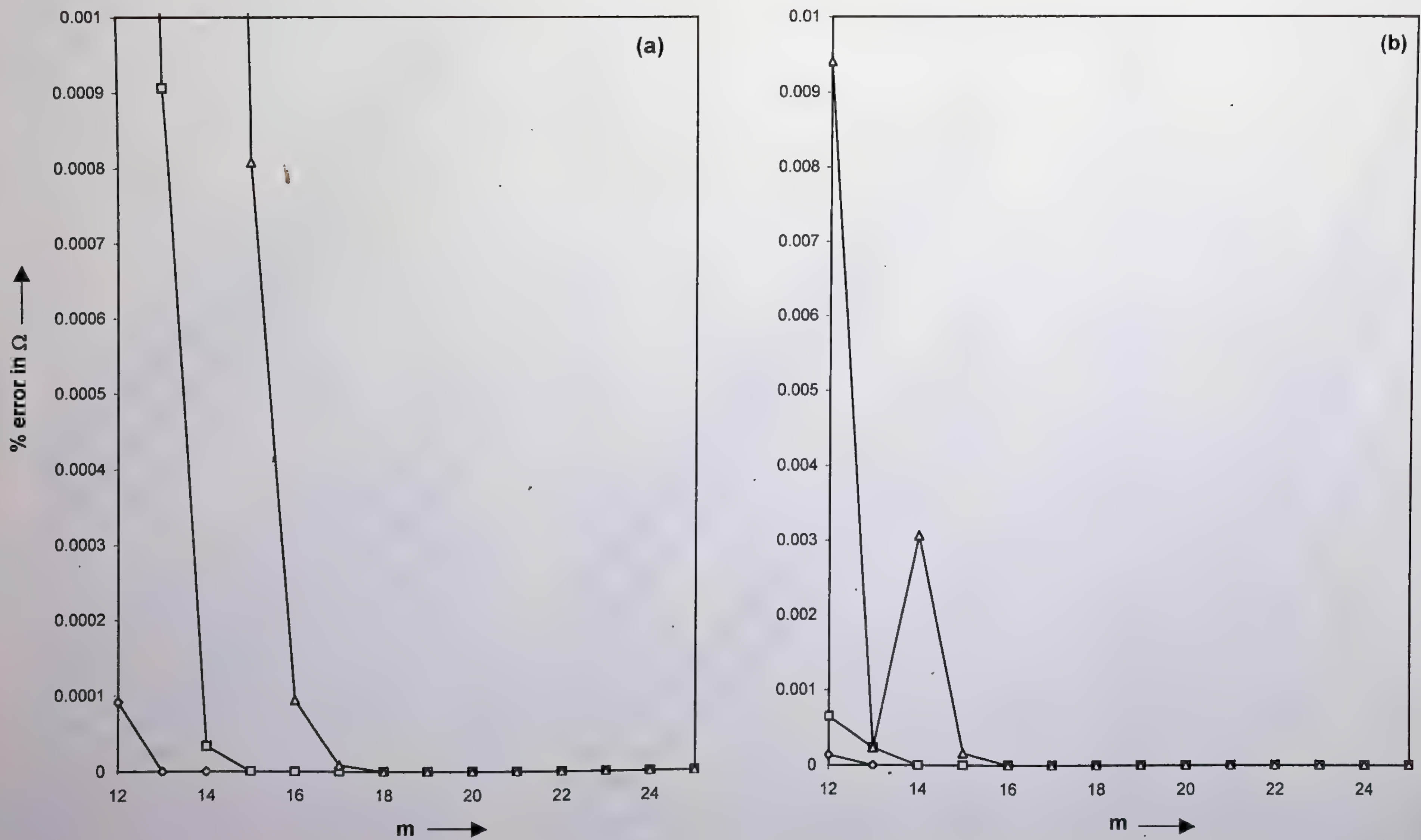

Fig. 8.1 : Convergence of frequency parameter for first three modes of vibrations for (a) C-C and (b) C-S plate for $\mu = 1.0$, $\eta = -0.5$, $n = 1$, $\varepsilon = 0.5$, $\alpha = 0.5$, $p = 0.5$, $K^{*} = 200$. $G = 25$. —◊—, fundamental mode; —□—, second mode; —Δ—, third mode..

Percentage error= $\left(\frac{|\Omega_{m} - \Omega_{25}|}{\Omega_{25}}\right) \times 100$

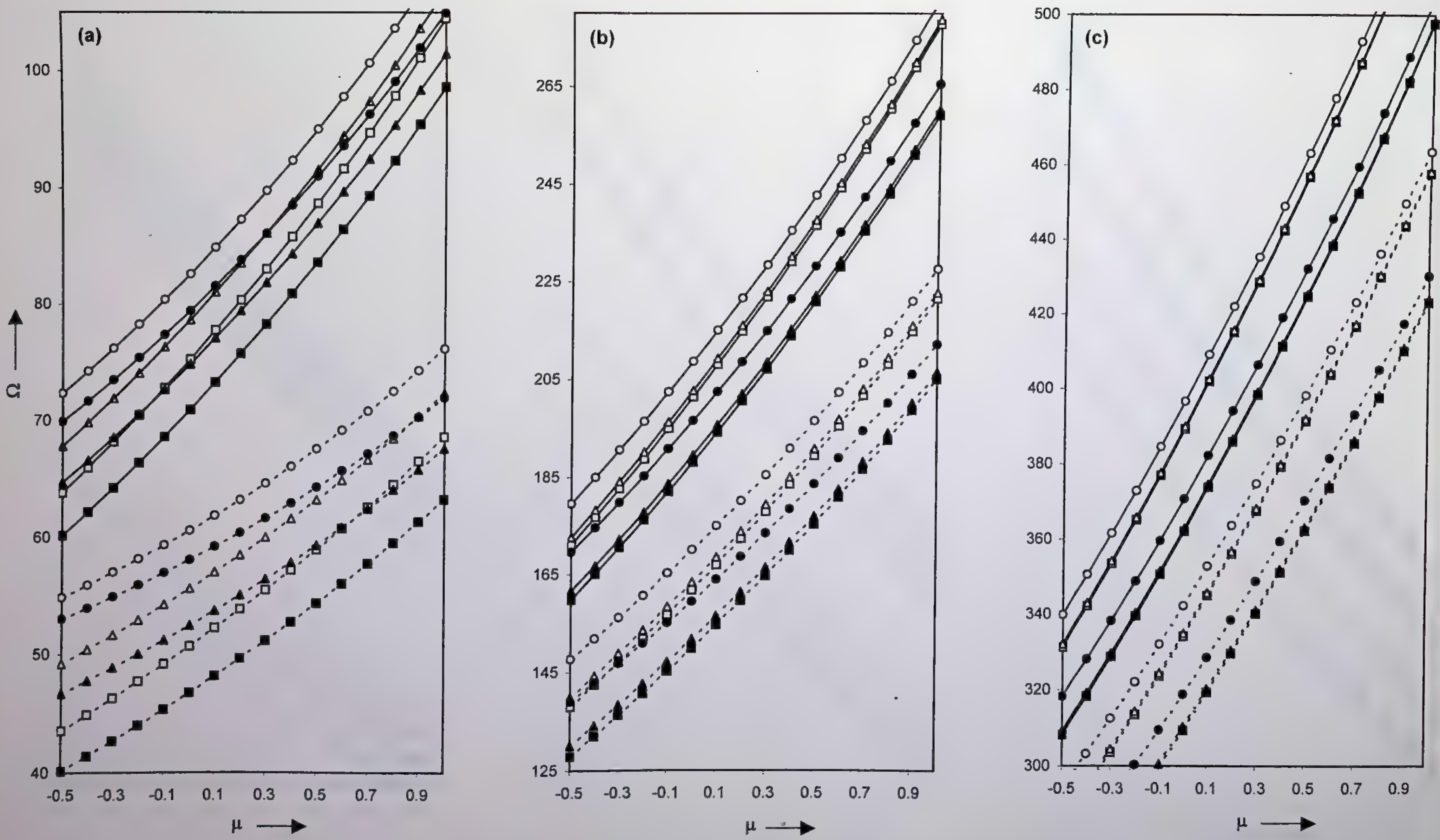

Fig. 8.2 : Frequency parameter for plates vibrating in (a) fundamental (b) second and (c) third mode for $\eta = -0.5$, $\alpha = 0.5$, $\varepsilon = 0.3$, $p = 5.0$.
————, C-C plate; - - - - - - - -, C-S plate.
□, $K^* = 0$, $G = 0$; Δ, $K^* = 500$, $G = 0$; ○, $K^* = 500$, $G = 25$. □, Δ, ○, $n = 1$; ■, ▲, ●, $n = 2$.

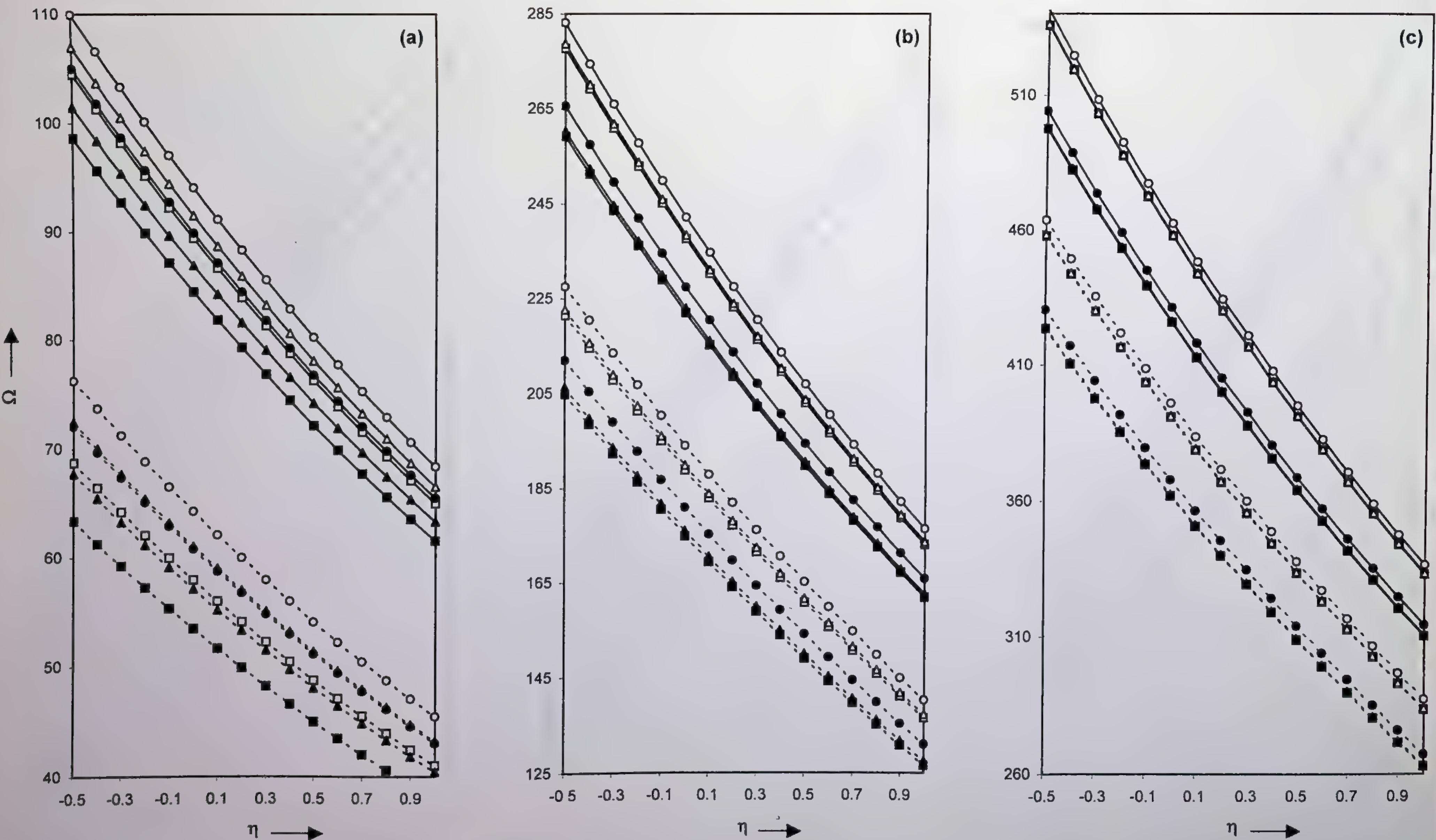

Fig. 8.3 : Frequency parameter for plates vibrating in (a) fundamental (b) second and (c) third mode for $\mu = 1.0$, $\alpha = 0.5$, $\varepsilon = 0.3$, $p = 5.0$.

————, C-C plate; -------------, C-S plate.

□, $K^* = 0$, $G = 0$; Δ, $K^* = 500$, $G = 0$; ○, $K^* = 500$, $G = 25$. □, Δ, ○, $n = 1$; ■, ▲, ●, $n = 2$.

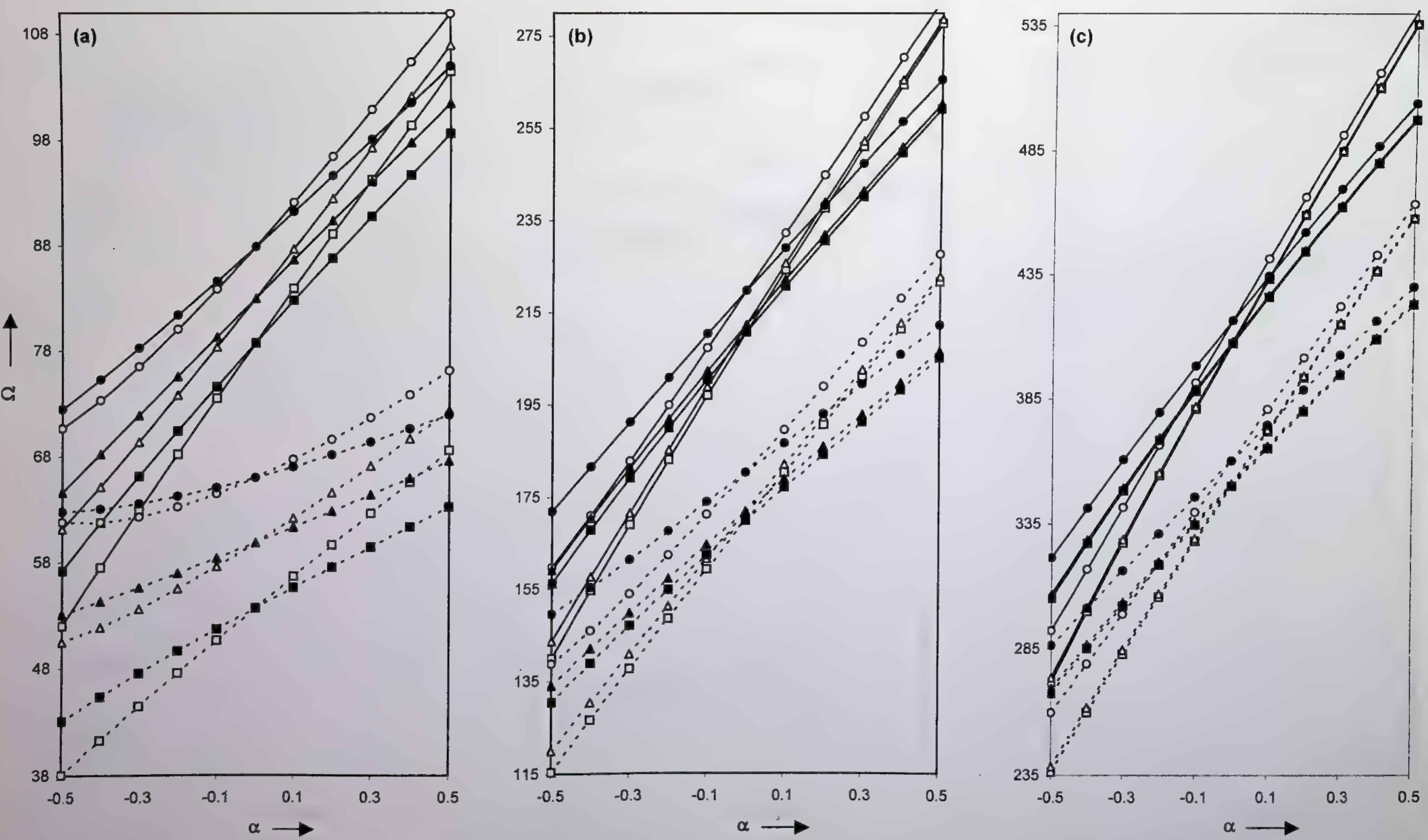

Fig. 8.4 : Frequency parameter for plates vibrating in (a) fundamental (b) second and (c) third mode for $\mu = 1.0$, $\eta = -0.5$, $\varepsilon = 0.3$, $p = 5.0$.
————, C-C plate; -------------, C-S plate.
□, $K^* = 0$, $G = 0$; Δ, $K^* = 500$, $G = 0$; ○, $K^* = 500$, $G = 25$. □, Δ, ○, $n = 1$; ■, ▲, ●, $n = 2$.

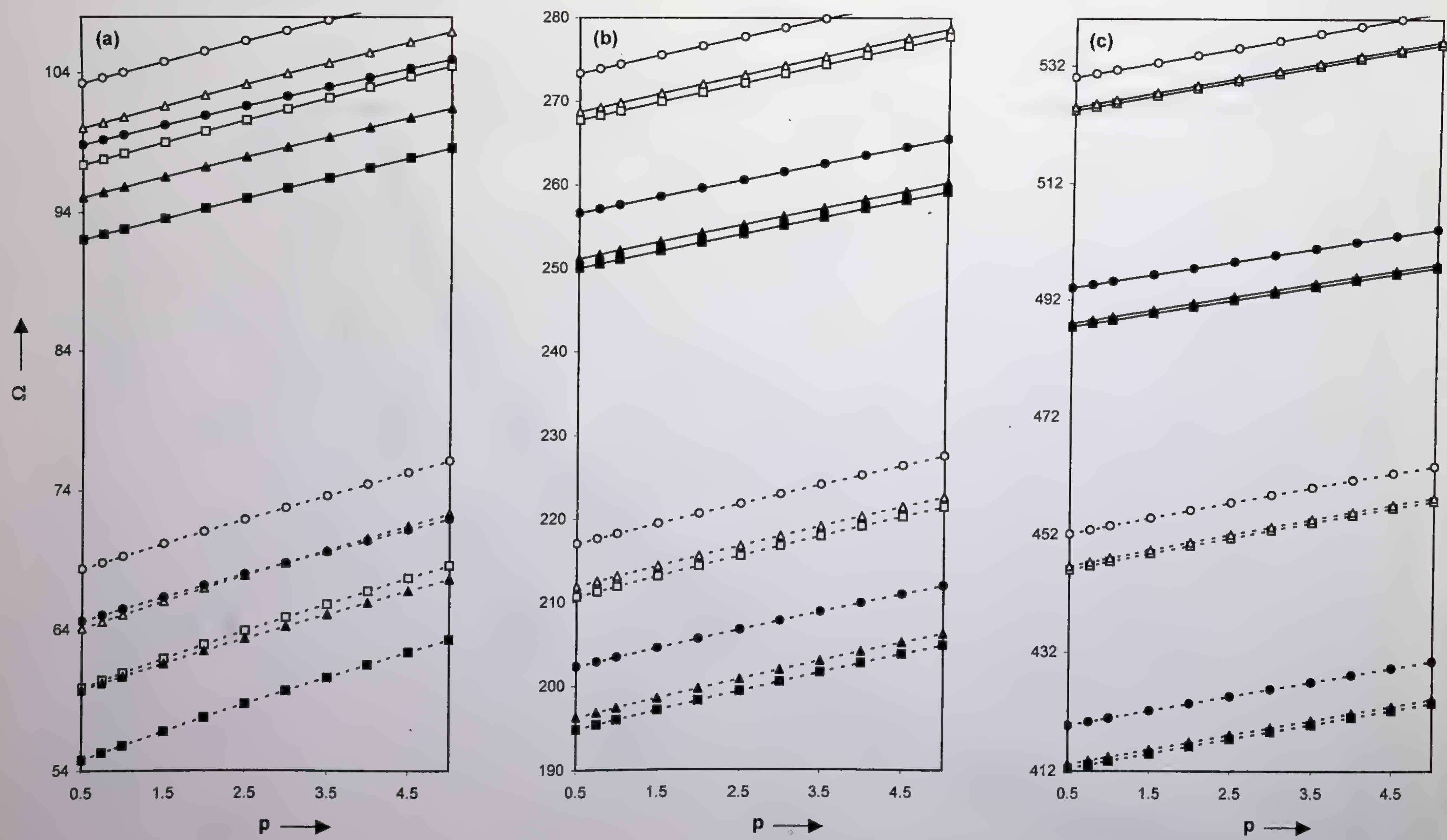

Fig. 8.5 : Frequency parameter for plates vibrating in (a) fundamental (b) second and (c) third mode for $\mu = 1.0$, $\eta = -0.5$, $\varepsilon = 0.3$, $\alpha = 0.5$.
———, C-C plate; -------------, C-S plate.
□, $K^* = 0$, $G = 0$; Δ, $K^* = 500$, $G = 0$; ○, $K^* = 500$, $G = 25$. □, Δ, ○, $n = 1$; ■, ▲, ●, $n = 2$.

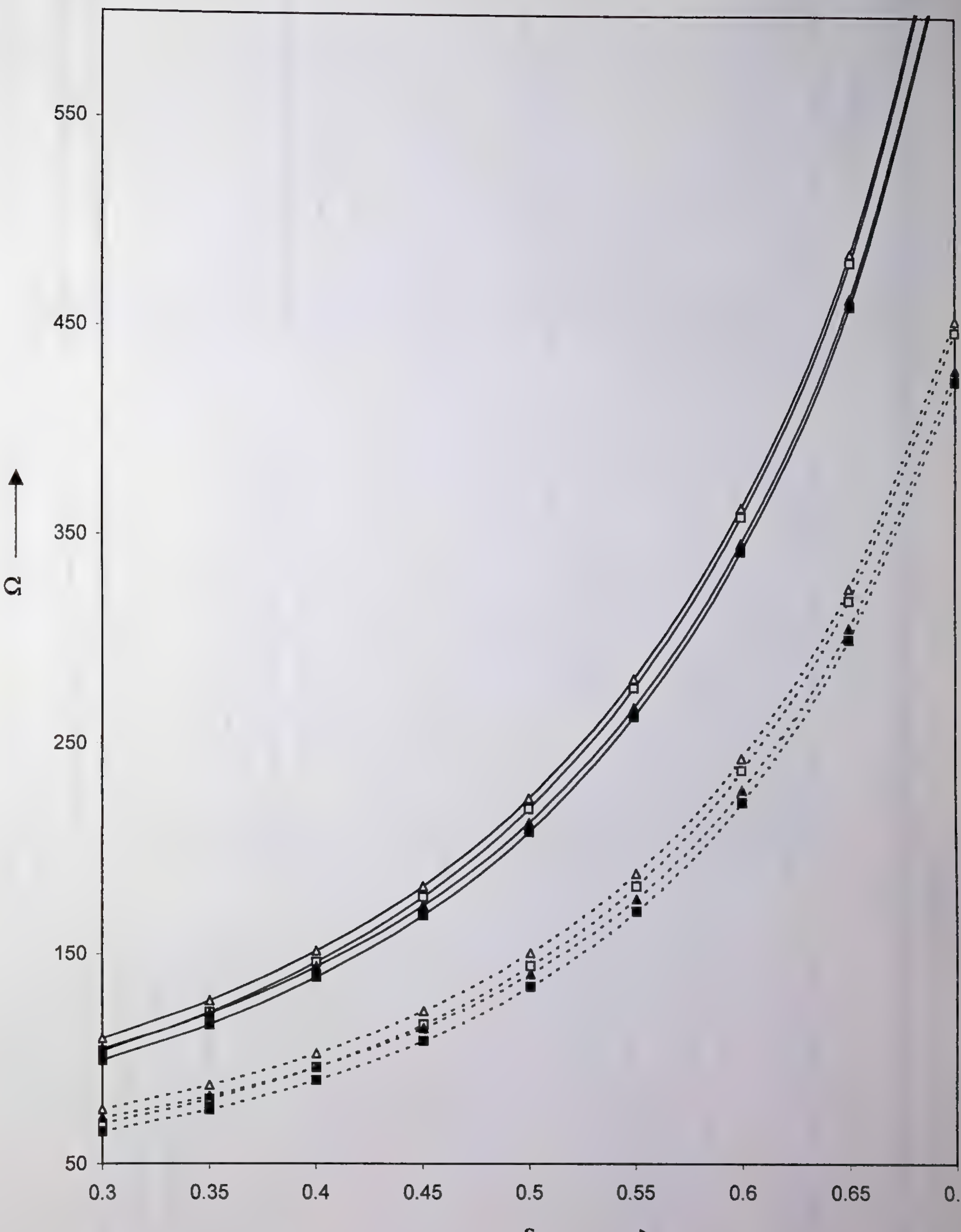

Fig. 8.6 : Frequency parameter for plates vibrating in fundamental mode for $\mu = 1.0$, $\eta = -0.5$
$\alpha = 0.5$, $K^{*} = 500$, $G = 25$. ————, C-C plate; -------------, C-S plate.
□, p = 1.0; Δ, p = 5.0. □, Δ, n = 1; ■, ▲, n = 2.

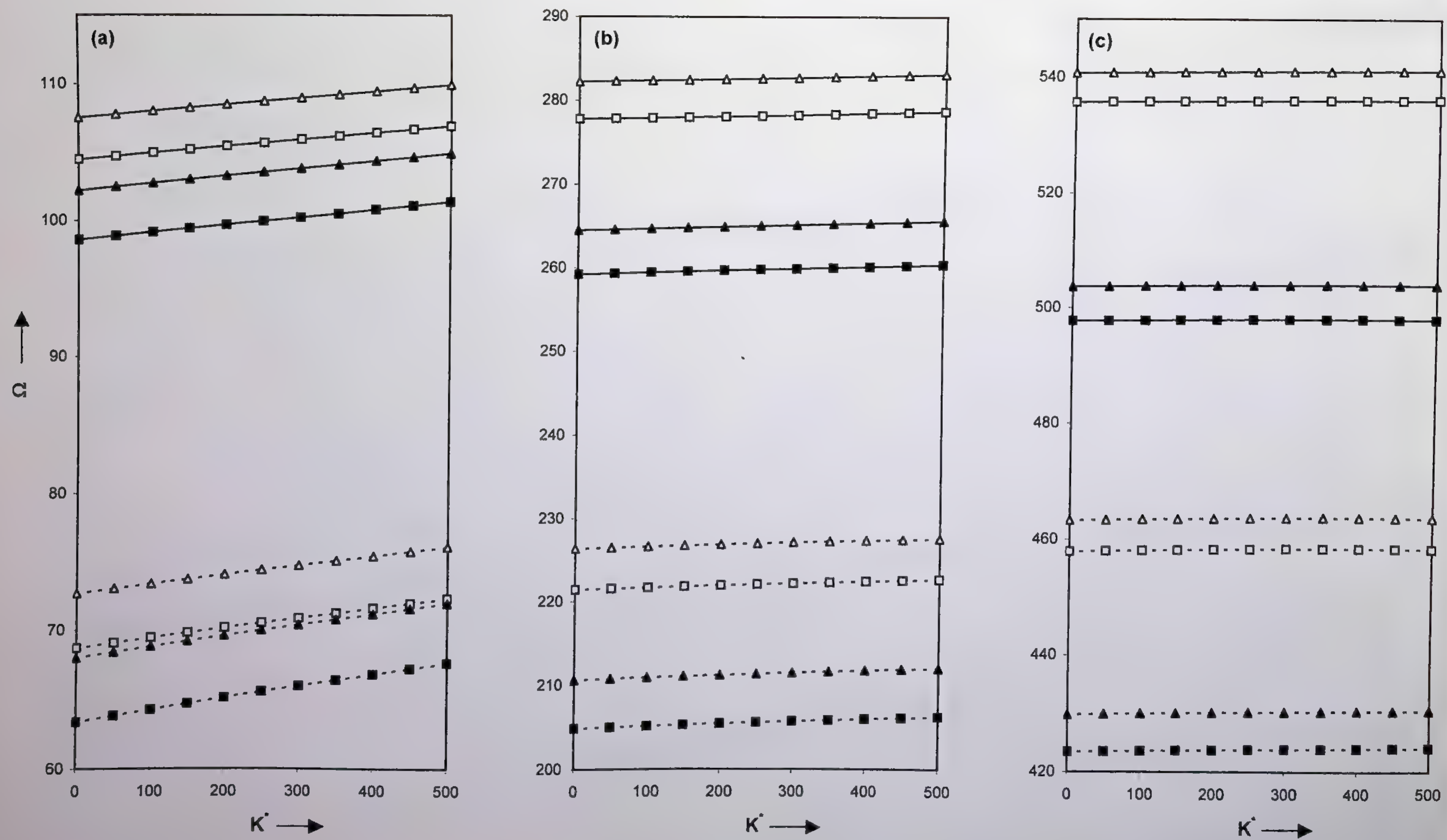

Fig. 8.7 : Frequency parameter for plates vibrating in (a) fundamental (b) second and (c) third mode for $\mu = 1.0$, $\eta = -0.5$, $\varepsilon = 0.3$, $\alpha = 0.5$, $p = 5.0$.
————, C-C plate; ------------, C-S plate.
□, G = 0; Δ, G = 25. □, Δ, n = 1; ■, ▲, n = 2.

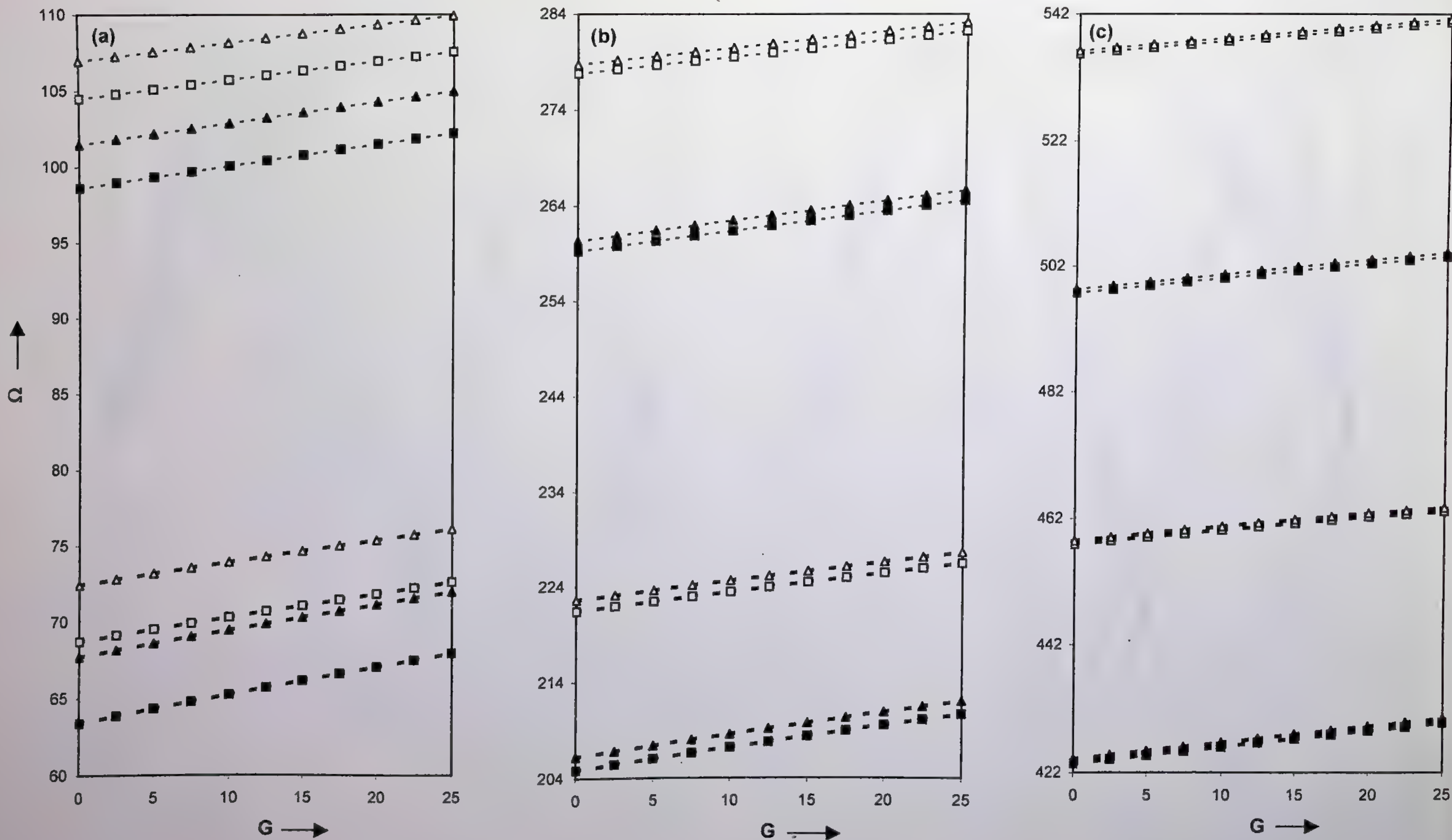

Fig. 8.8 : Frequency parameter for plates vibrating in (a) fundamental (b) second and (c) third mode for $\mu = 1.0$, $\eta = -0.5$, $\varepsilon = 0.3$, $\alpha = 0.5$, $p = 5.0$.
————, C-C plate; -------------, C-S plate.
□, $K^* = 0$; Δ, $K^* = 500$. □, Δ, $n = 1$; ■, ▲, $n = 2$.

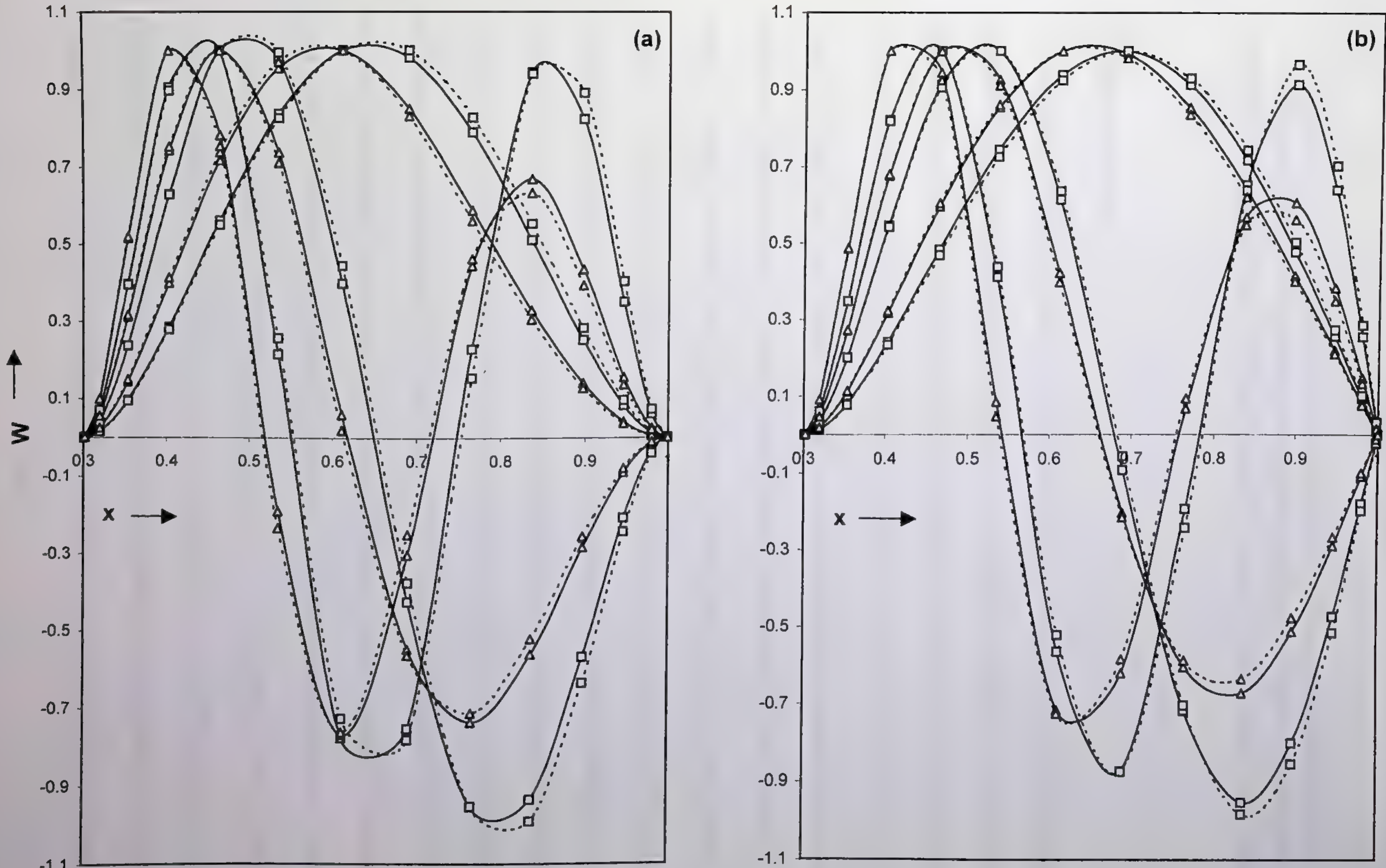

Fig. 8.9 : Normalized displacements for (a) C-C plate and (b) C-S plate for $\mu = 1.0$, $\eta = -0.5$, $p = 5.0$, $K^* = 200$, $G = 25$, ε=0.3.
————, n = 1; ----------, n = 2. □, $\alpha = -0.5$; ■, $\alpha = 0.5$.

CHAPTER IX

EFFECT OF THERMAL GRADIENT ON AXISYMMETRIC VIBRATIONS OF NON-UNIFORM POLAR ORTHOTROPIC CIRCULAR PLATES WITH ELASTICALLY RESTRAINED EDGE

1. INTRODUCTION

The analysis of vibration of plates with elastically restrained edge is an important problem in aeronautical and naval structural engineering. In aircraft structures, individual plates are connected to other plates or stiffeners at their boundaries and thus they have elastic restraint at their edges. Thermally induced vibrations of non-uniform polar orthotropic circular plates are of great interest in air-craft, machine design and also in nuclear, astronautical and chemical engineering. In the presence of thermal gradient, elastic coefficients become functions of space variable, causing non-homogeneity in the material. Most of the engineering materials are found to have a linear relationship between modulus of elasticity and temperature (Nowacki[1962], Fauconneau and Marangoni[1970]). Due to the increasing use of modern materials in structural components, various researchers have carried out a number of studies dealing with the effect of thermal gradient on vibration of isotropic/orthotropic plates of various geometries with uniform/non-uniform thickness. Irie and Yamada[1978] studied thermally induced vibration of elastically supported circular and annular plates. Ganesan and Dhotarad[1979] analysed the influence of thermal gradient on natural frequencies of tapered orthotropic plates. Gupta[1984] and Tomar and Gupta[1984a,1984b] investigated the effect of thermal gradient on non-uniform polar orthotropic circular and elliptic plates respectively. Gorman[1985a, 1985b] analysed thermal gradient effects upon the vibrations of composite circular plates. Rao et al.[1996] presented vibration analysis of a thermally stressed spinning plate using finite element method.

Li and Zhou[2001] used shooting method for non-linear vibration and thermal buckling of heated orthotropic circular plates. Arafat et al.[2004] analysed the vibration of circular and annular plates with clamped edges subjected to in-plane thermal loads.

In this chapter, axisymmetric vibrations of polar orthotropic circular plates of quadratically varying thickness with restrained elastic edge subjected to constant thermal gradient have been discussed on the basis of classical plate theory. Ritz method has been employed to obtain approximate solution of the problem, where basis functions based upon the static deflection for isotropic plates have been used. The choice of this method has the advantage of high accuracy and computational efficiency, which greatly depends upon the nature of admissible functions. The consideration of thermal gradient causes non-homogeneity i.e. variation in mechanical properties of plate material. This variation has been taken into account by assuming that Young's moduli of the plate vary linearly with radius vector. The first three natural frequencies have been obtained for different values of flexibility conditions and for classical edge conditions: clamped, simply supported and free. The effect of edge conditions and that of orthotropy, thermal gradient and thickness variation on the natural frequencies has been investigated for the first three modes of vibration. Normalised displacements for specified plate parameters have been drawn for all the plates. Results for linear as well as parabolic thickness variation have been obtained as special cases. Comparison studies have been carried out which establish the accuracy of present method.

2. BASIC PLATE EQUATION

Consider a thin circular plate of radius a, thickness $h(r)$, density ρ, elastically restrained against translation and rotation by springs of stiffness k and k_φ, referred to cylindrical polar coordinates

(r, θ, z), where the axis of the plate is taken as the line $r = 0$ and its middle surface as the plane $z = 0$. Let the plate be subjected to a steady one-dimensional temperature distribution T.

Energy expressions

The strain-displacement relations, stress-strain relations and moment resultants per unit length are obtained following equations (6.2.1)-(6.2.3).

The total kinetic energy which results from vertical displacement of the elements of the plate, is given by

$$T = \frac{1}{2}\rho \int_0^{2\pi}\int_0^a\int_{-h/2}^{h/2} \left(\frac{\partial w}{\partial t}\right)^2 r\, dz\, dr\, d\theta \quad . \tag{9.2.1}$$

Integrating with respect to z, we get

$$T = \frac{1}{2}\rho \int_0^{2\pi}\int_0^a h\left(\frac{\partial w}{\partial t}\right)^2 r\, dr\, d\theta \quad . \tag{9.2.2}$$

The bending strain energy of the plate is defined by

$$U_B = \frac{1}{2}\int_0^{2\pi}\int_0^a\int_{-h/2}^{h/2} (\sigma_r\varepsilon_r + \sigma_\theta\varepsilon_\theta) r\ dz\, dr\, d\theta \ .$$

Substituting the values of ε_r, ε_θ and σ_r, σ_θ from equations (4.2.2) and (6.2.1), we get

$$U_B = \frac{1}{2}\int_0^{2\pi}\int_0^a \left[D_r\left\{\left(\frac{\partial^2 w}{\partial r^2}\right)^2 + 2\upsilon_\theta \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r}\right)\right\} + D_\theta\left(\frac{1}{r}\frac{\partial w}{\partial r}\right)^2\right] r\, dr d\theta \ , \tag{9.2.3}$$

where, $D_r = \dfrac{E_r h^3}{12(1-\upsilon_r\upsilon_\theta)}$, $D_\theta = \dfrac{E_\theta h^3}{12(1-\upsilon_r\upsilon_\theta)}$ are flexural rigidities of the plate, E_r, E_θ, υ_r, υ_θ are respectively the Young's moduli and Poisson's ratios of the plate material in the proper directions with $\upsilon_r E_\theta = E_r \upsilon_\theta$.

In the case of elastic restraint against rotation at the boundary, the strain energy U_{ER} stored in the plate during vibration is given by(Szilard[1974], pp 219)

$$U_{ER} = \frac{1}{2}\oint \left(\frac{\partial w}{\partial n}\right)^2 k_\varphi ds,$$

where, $1/k_\varphi$ is the rotational flexibility of the springs and n represents the outward normal to the boundary. The above expression can be written as

$$U_{ER} = \frac{1}{2} a k_\varphi \int_0^{2\pi} \left(\frac{\partial w}{\partial r}\right)^2_{r=a} d\theta \ . \tag{9.2.4}$$

The strain energy due to elastic restraint against translation at the boundary is

$$U_{ET} = \frac{1}{2}\oint w^2 k\, ds \quad ,$$

where, k is the translational flexibility of the springs, which becomes

$$U_{ET} = \frac{1}{2} a k \int_0^{2\pi} w^2(a,\theta)\, d\theta \ . \tag{9.2.5}$$

The total potential energy of the plate is $U = U_B + U_{ER} + U_{ET}$. Thus,

$$\begin{aligned} U = \frac{1}{2}\int_0^{2\pi}\int_0^a \left[D_r \left\{ \left(\frac{\partial^2 w}{\partial r^2}\right)^2 + 2\upsilon_\theta \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r}\right) \right\} + D_\theta \left(\frac{1}{r}\frac{\partial w}{\partial r}\right)^2 \right] r\, dr\, d\theta \\ + \frac{1}{2} a k_\varphi \int_0^{2\pi} \left(\frac{\partial w(a,\theta)}{\partial r}\right)^2 d\theta + \frac{1}{2} a k \int_0^{2\pi} w^2(a,\theta)\, d\theta. \end{aligned} \tag{9.2.6}$$

For harmonic vibrations, the transverse deflection can be written as

$$w(r,t) = W(r)\sin\omega t \ , \tag{9.2.7}$$

where, $W(r)$ is the shape function and ω is the circular frequency in radians per second. Substituting for w from equation (9.2.7) in (9.2.2), the kinetic energy will be

$$T = \frac{1}{2}\rho\omega^2\cos^2\omega t\int_0^{2\pi}\int_0^a hW^2 r\,dr\,d\theta \ .$$

Hence, the maximum kinetic energy of the plate is given by

$$T_{\max} = \frac{1}{2}\rho\omega^2\int_0^{2\pi}\int_0^a hW^2 r\,dr\,d\theta \ . \tag{9.2.8}$$

Similarly, the maximum potential energy of the plate is given by

$$\begin{aligned} U_{\max} = \frac{1}{2}\int_0^{2\pi}\int_0^a &\left[D_r\left\{\left(\frac{\partial^2 W}{\partial r^2}\right)^2 + 2\upsilon_\theta\frac{\partial^2 W}{\partial r^2}\left(\frac{1}{r}\frac{\partial W}{\partial r}\right)\right\} + D_\theta\left(\frac{1}{r}\frac{\partial W}{\partial r}\right)^2\right] r\,dr\,d\theta \\ &+\frac{1}{2}ak_\varphi\int_0^{2\pi}\left(\frac{\partial W(a,\theta)}{\partial r}\right)^2 d\theta + \frac{1}{2}ak\int_0^{2\pi}W^2(a,\theta)\,d\theta. \end{aligned} \tag{9.2.9}$$

3. METHOD OF SOLUTION : RITZ METHOD

Ritz method requires that the functional

$$\begin{aligned} J(W) &= U_{\max} - T_{\max} \\ &= \frac{1}{2}\int_0^{2\pi}\int_0^a\left[D_r\left\{\left(\frac{\partial^2 W}{\partial r^2}\right)^2 + 2\upsilon_\theta\frac{\partial^2 W}{\partial r^2}\left(\frac{1}{r}\frac{\partial W}{\partial r}\right)\right\} + D_\theta\left(\frac{1}{r}\frac{\partial W}{\partial r}\right)^2\right] r\,dr\,d\theta \\ &\quad+\frac{1}{2}ak_\varphi\int_0^{2\pi}\left(\frac{\partial W(a,\theta)}{\partial r}\right)^2 d\theta + \frac{1}{2}ak\int_0^{2\pi}W^2(a,\theta)\,d\theta - \frac{1}{2}\rho\omega^2\int_0^{2\pi}\int_0^a hW^2 r\,dr\,d\theta, \end{aligned} \tag{9.3.1}$$

be minimized.

Now, we approximate transverse deflection W in terms of a set of linearly independent coordinate functions, which satisfy boundary conditions of the problem. The choice of basis functions to approximate the deflection using Ritz method has its own significance. The deflection function assumed here is based upon the static deflection for isotropic circular plates.

Introducing non-dimensional variables $\overline{W} = \frac{W}{a}$, $R = \frac{r}{a}$, and considering the thickness variation as $h = h_0\left(1 + \alpha R + \beta R^2\right)$, the temperature distribution is given by

$$T = T_0\left(1 - R\right), \tag{9.3.2}$$

where h_0 is the thickness of the plate at its centre, T is the temperature excess above the reference temperature at any point R, T_0 is the temperature excess at the centre $R = 0$ above the reference temperature at any point on the boundary of the plate. For most engineering materials, the temperature dependence of the modulus of elasticity is given by a relation of the type

$$E_r(T) = E_1\left(1 - \gamma T\right) \text{ and } E_\theta(T) = E_2\left(1 - \gamma T\right). \tag{9.3.3}$$

Using (9.3.2), relations (9.3.3) reduce to

$$E_r(T) = E_1\left(1 - \zeta(1 - R)\right) \text{ and } E_\theta(T) = E_2\left(1 - \zeta(1 - R)\right). \tag{9.3.4}$$

Assume the deflection function as

$$\overline{W} = \sum_{i=0}^{m} A_i F_i(R) = \sum_{i=0}^{m} A_i\left(1 + \alpha_i R^4 + \beta_i R^2\right) R^{2i} , \tag{9.3.5}$$

where, A_i are undetermined coefficients , α_i, β_i are unknown constants.

As each co-ordinate function has to satisfy elastically restrained against rotation and translation conditions at the boundary, then

$$K_\varphi \frac{d\overline{W}(1)}{dR} = -(1+\alpha+\beta)^3 \left[\frac{d^2\overline{W}}{dR^2} + \upsilon_\theta \left(\frac{1}{R}\frac{d\overline{W}}{dR} \right) \right]_{R=1} , \tag{9.3.6}$$

$$K\overline{W}(1) = (1+\alpha+\beta)^3 \left[\frac{d}{dR}\left(\frac{d^2\overline{W}}{dR^2} + \frac{1}{R}\frac{d\overline{W}}{dR} \right) \right]_{R=1} . \tag{9.3.7}$$

The unknown constants α_i, β_i are determined using these boundary conditions, which give

$$\alpha_i = \frac{s_{23}s_{12} - s_{13}s_{22}}{s_{22}s_{11} - s_{21}s_{12}}, \qquad \beta_i = \frac{s_{13}s_{21} - s_{23}s_{11}}{s_{22}s_{11} - s_{21}s_{12}},$$

where

$$s_{11} = (2i+4)K_\varphi + (1+\alpha+\beta)^3(2i+4)(2i+3+\upsilon_\theta) ,$$

$$s_{12} = (2i+2)K_\varphi + (1+\alpha+\beta)^3(2i+2)(2i+1+\upsilon_\theta) ,$$

$$s_{13} = 2iK_\varphi + (1+\alpha+\beta)^3 2i(2i-1+\upsilon_\theta) ,$$

$$s_{21} = K - (1+\alpha+\beta)^3(2i+4)^2(2i+2) ,$$

$$s_{22} = K - (1+\alpha+\beta)^3(2i+2)^2 2i ,$$

$$s_{23} = K - (1+\alpha+\beta)^3(2i)^2(2i-2) ,$$

where $K = \dfrac{a^3k}{D_{r_0}}$, $\quad K_\varphi = \dfrac{ak_\varphi}{D_{r_0}}$.

Using non-dimensional variables $\overline{W}$ and R along with the relations (9.3.4) and (9.3.5), the functional $J(W)$ given by equation (9.3.1) becomes

$$J(\overline{W})=\frac{D_{r_0}}{2}\left[\int_0^{2\pi}\int_0^1\left[\left(1+\alpha R+\beta R^2\right)^3(\zeta_1+\zeta R)\left\{\left(\frac{\partial^2\overline{W}}{\partial R^2}\right)^2+\frac{2\upsilon_\theta}{R}\frac{\partial^2\overline{W}}{\partial R^2}\frac{\partial\overline{W}}{\partial R}\right.\right.\right.$$
$$\left.\left.+p^2\left(\frac{1}{R}\frac{\partial\overline{W}}{\partial R}\right)^2\right\}\right]R\,d\theta\,dR+K\int_0^{2\pi}\overline{W}^2(1)d\theta+K_\varphi\int_0^{2\pi}\left(\frac{\partial\overline{W}(1)}{\partial R}\right)^2d\theta \tag{9.3.8}$$
$$\left.-\Omega^2\int_0^1\int_0^{2\pi}\left(1+\alpha R+\beta R^2\right)\overline{W}^2R\,d\theta\,dR\right],$$

where $D_{r_0}=\dfrac{E_r h_0^3}{12(1-\upsilon_r\upsilon_\theta)}$, $\zeta_1=1-\zeta$, $p^2=\dfrac{E_\theta}{E_r}$, $\Omega^2=\dfrac{a^4\omega^2\rho h_0}{D_{r_0}}$.

The minimization of the functional $J(\overline{W})$ given by (9.3.8) requires

$$\frac{\partial J(\overline{W})}{\partial A_i}=0,\quad i=0,1,2,\ldots,m. \tag{9.3.9}$$

This leads to a system of homogeneous equations in A_i, $i = 0, 1,\ldots, m$, whose non-trivial solution leads to the frequency equation

$$\left|A-\Omega^2B\right|=0, \tag{9.3.10}$$

where, $A=[a_{ij}]$ and $B=[b_{ij}]$ are square matrices of order $(m+1)$ given by

$$a_{ij}=\int_0^1\left(1+\alpha R+\beta R^2\right)^3(\zeta_1+\zeta R)\left[F_i''F_j''+\frac{\upsilon_\theta}{R}\left(F_i''F_j'+F_j''F_i'\right)+\frac{p^2}{R^2}F_i'F_j'\right]R\,dR$$
$$+K_\varphi F_i'(1)F_j'(1)+K\,F_i(1)F_j(1) \tag{9.3.11}$$

and $$b_{ij}=\int_0^1\left(1+\alpha R+\beta R^2\right)F_iF_jR\,dR, \tag{9.3.12}$$

for $i=0,1,\ldots,m$; $j=0,1,\ldots,m$.

4. NUMERICAL RESULTS AND DISCUSSIONS

The frequency equation (9.3.10) has been solved to obtain first three natural frequencies for various values of plate parameters, such as rigidity ratio p (= 0.5, 0.75, 1.0, 2.0, 3.0, 4.0, 5.0), thermal gradient ζ (= 0.0(0.1)0.5), taper parameters α (= -0.5(0.1)0.5) ; β (= -0.5(0.1)0.5) such that $\alpha + \beta > -1.0$, flexibility parameters K (= 0, 10, 100, $10^{20} \cong \infty$); K_φ (= 0, 10, 100, $10^{20} \cong \infty$) and $\upsilon_\theta = 0.3$.

All natural frequencies obtained from Ritz method are upper bounds of the exact ones and therefore, convergence should be monotonic from above as the number of terms of admissible functions increases. The convergence study has been carried out for circular plates with $\upsilon_\theta = 0.3$ for different sets of plate parameters. The convergence graphs for clamped, simply supported and free plates are shown in Figures 9.1(a,b,c) for $\zeta = 0.5$, $\alpha = -0.3$, $\beta = -0.2$, $p = 5.0$. It is observed that 11 terms of admissible function give first three frequency parameters at least accurate to four significant digits.

Numerical results are presented in Tables (9.1-9.7) and Figures (9.2-9.5). Table 9.1 gives the value of frequency parameter Ω for different values of flexibility parameter K_φ, rigidity ratio p, thermal gradient ζ for Linearly Varying Thickness (LVT), Parabolically Varying Thickness (PVT) and Quadratically Varying Thickness (QVT) plates for the first three modes of vibration, when the stiffness of translational spring $K = 0$. The free edge classical boundary condition corresponds to flexibility parameters $K = 0$ and $K_\varphi = 0$. The frequency parameter Ω is found to increase with the increase in rigidity ratio p as well as flexibility parameter K_φ, keeping all the other plate parameters fixed. The effect of flexibility parameter K_φ is more pronounced in range of zero to ten in all the three modes of vibration. The frequency parameter Ω decreases with the

Ω decreases with the increasing value of thermal gradient ζ. Further, Ω is found to increase with increasing values of taper parameters α and β. The frequency parameter $\Omega_{LVT} > \Omega_{PVT}$ for positive values of taper parameters α and β, while $\Omega_{PVT} > \Omega_{LVT}$ for negative values of α and β. Tables 9.2-9.4 present the frequency parameter Ω for $K = 10$, $K = 100$ and $K = 10^{20}$ respectively, other plate parameters being the same as in table 9.1. Also, the frequency parameter Ω for clamped plate ($K_\varphi = 10^{20}$ and $K = 10^{20}$) is greater than that for the simply supported plate ($K_\varphi = 0$ and $K = 10^{20}$) presented in table 9.4. Tables (9.5-9.7) present the value of frequency parameter Ω for $p = 0.5, 1.0, 2.0$, $\zeta = 0.0, 0.1, 0.2, 0.3$, $\alpha = -0.5, 0.0, 0.5$, $\beta = -0.5, 0.0, 0.5$ for clamped, simply supported and free plates respectively. It is seen that the frequency parameter Ω for free plate is greater than that for simply supported plate and lesser than that for clamped plate for $\alpha > 0, \beta > 0$.

Figures 9.2(a,b,c) show the plots for frequency parameter Ω versus thermal gradient ζ for $\alpha = 0.5$, $\beta = 0.5$ and different values of rigidity ratio $p(= 0.5, 1.0, 2.0)$ for clamped, simply supported and free plate for the first three modes of vibration, respectively. It is observed that frequency parameter Ω decreases with increasing value of thermal gradient ζ. It can be seen that the effect of orthotropy decreases in the order of plates free, simply supported and clamped.

Figures 9.3(a,b,c) depict the variation of frequency parameter Ω versus taper parameter α for $\zeta = 0.5$, $p = 5.0$, $\beta = 0.0, 0.5$ for clamped, simply supported and free plate for the first three modes of vibrations, respectively. It is found that frequency increases with increasing value of α. Figures 9.4(a,b,c) show the behaviour of frequency parameter Ω with taper parameter β. It is observed that frequency increases with increasing value of β. The rate of increase of Ω with α

as well as β for free plate is higher than that for simply supported plate and less than that for clamped plate. Also, the frequency for linearly as well as parabolically tapered plate is smaller than that for quadratically tapered plate for the same values of taper parameters.

Figures 9.5(a,b,c) show the plots for Normalized displacements for $\zeta = 0.0,\ 0.5,\ \alpha = 0.5,$ $\beta = 0.5,\ p = 5.0$ for the first three modes of vibration for clamped, simply supported and free plates respectively. It is observed that the effect of thermal gradient decreases the radii of nodal circles.

A comparison of results has been presented in Table 9.8 and 9.9 for uniform isotropic circular plate with exact solution given by Leissa[1969] and approximate solutions obtained by Azimi[1988] using receptence method, Ansari[2000] by using Ritz method, Pardoen[1978] employing finite element method. Table 9.10 shows the comparison of results for parabolically tapered orthotropic plate with those of Ansari[2000] by Ritz method. Comparison of the results of this study for linearly and parabolically tapered orthotropic plates subjected to thermal gradient with those obtained by Gupta[1984] using Frobenius method has been given in table 9.11. A close agreement of results shows the accuracy and versatility of the present method.

Table 9.1
Values of frequency parameter Ω for K = 0.0

		α	-0.5		0.0			0.5		
K_ϕ	p	ζ \ β	0.0	0.5	-0.5	0.0	0.5	-0.5	0.0	0.5
					I					
0	0.5	0.1	6.1357	6.3632	6.9699	7.0904	7.5682	7.8144	8.2259	8.8480
		0.2	5.9250	6.1534	6.7254	6.8520	7.3321	7.5480	7.9639	8.5887
		0.3	5.7039	5.9336	6.4680	6.6024	7.0853	7.2677	7.6904	8.3186
	5	0.1	11.3841	14.2137	13.3747	15.9993	19.1964	17.8336	20.9105	24.2773
		0.2	11.1297	13.9171	13.0709	15.6583	18.8084	17.4470	20.4802	23.7968
		0.3	10.8661	13.6097	12.7560	15.3051	18.4061	17.0467	20.0345	23.2983
10	0.5	0.1	8.8050	11.1432	9.5891	11.8211	13.3259	12.4532	13.8069	14.6078
		0.2	8.5723	10.8947	9.3314	11.5587	13.0714	12.1757	13.5420	14.3587
		0.3	8.3274	10.6319	9.0596	11.2821	12.8018	11.8819	13.2628	14.0958
	5	0.1	13.9781	17.7396	15.9685	19.5796	22.5904	21.4053	24.2801	27.0515
		0.2	13.6793	17.3779	15.6239	19.1808	22.1502	20.9678	23.8054	26.5362
		0.3	13.3680	16.9996	15.2646	18.7643	21.6894	20.5109	23.3098	25.9980
100	0.5	0.1	8.9259	11.9216	9.7427	12.7815	15.7555	13.5943	16.5906	19.3351
		0.2	8.6895	11.6433	9.481	12.4871	15.4172	13.2836	16.2414	18.95
		0.3	8.4408	11.3497	9.2049	12.1771	15.0595	12.9547	15.8734	18.5421
	5	0.1	14.1282	18.6236	16.1572	20.6533	24.9761	22.6601	26.9744	30.9834
		0.2	13.8244	18.2279	15.8067	20.2168	24.4511	22.1821	26.4124	30.3444
		0.3	13.5079	17.8143	15.4415	19.7611	23.9013	21.6828	25.8246	29.6745
10^{20}	0.5	0.1	8.9400	12.0238	9.7609	12.9128	16.1584	13.7568	17.0839	20.3990
		0.2	8.7032	11.7412	9.4987	12.6133	15.8018	13.4404	16.7139	19.9644
		0.3	8.4540	11.4430	9.2220	12.2980	15.4251	13.1051	16.3244	19.5051
	5	0.1	14.1460	18.7490	16.1798	20.8125	25.4574	22.8548	27.5563	32.2057
		0.2	13.8416	18.3479	15.8286	20.3697	24.9105	22.3695	26.9693	31.5102
		0.3	13.5244	17.9288	15.4627	19.9073	24.3382	21.8625	26.3558	30.7818
					II					
0	0.5	0.1	26.2110	31.0602	30.1177	34.8682	39.3163	38.7235	43.1746	47.4216
		0.2	25.5066	30.1808	29.3180	33.8937	38.1847	37.6527	41.9429	46.0391
		0.3	24.7674	29.2568	28.4755	32.8706	36.9941	36.5211	40.6502	44.5843
	5	0.1	35.7669	43.9117	41.2654	49.3127	56.8295	54.7615	62.2798	69.4638
		0.2	34.9068	42.8054	40.2728	48.0692	55.3634	53.3807	60.6685	67.6384
		0.3	34.0077	41.6471	39.2326	46.7674	53.8249	51.9303	58.9810	65.7222
10	0.5	0.1	31.8598	38.5589	35.5940	42.0280	45.8619	45.5612	49.3148	52.3168
		0.2	31.0518	37.5840	34.6914	40.9627	44.6913	44.4026	48.0444	50.9284
		0.3	30.2014	36.5538	33.7355	39.8385	43.4542	43.1707	46.7060	49.4638
	5	0.1	42.1004	51.4354	47.5115	56.6141	62.8795	61.8345	68.0537	73.7569
		0.2	41.1113	50.2090	46.3894	55.2548	61.3506	60.3411	66.3811	71.9027
		0.3	40.0737	48.9181	45.2096	53.8255	59.7395	58.7649	64.6236	69.9516

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100	0.5	0.1	32.2645	41.0307	36.0731	44.801	52.0111	48.6078	55.7826	61.6116
		0.2	31.4423	39.9694	35.1543	43.6446	50.6683	47.3538	54.3406	60.0284
		0.3	30.5771	38.8484	34.1808	42.4244	49.2467	46.0189	52.8183	58.3507
	5	0.1	42.6143	54.3902	48.1211	59.9301	69.6342	65.4807	75.1864	83.2107
		0.2	41.6071	53.06	46.9781	58.4596	67.9081	63.8694	73.3126	81.1326
		0.3	40.5507	51.6599	45.7759	56.9133	66.0848	62.1671	71.34	78.9368
10^{20}	0.5	0.1	32.3129	41.3968	36.1314	45.2332	53.3796	49.1070	57.3375	64.9821
		0.2	31.4889	40.3205	35.2105	44.0597	51.9801	47.8337	55.8325	63.2615
		0.3	30.6218	39.1837	34.2348	42.8216	50.4987	46.4767	54.2444	61.4371
	5	0.1	42.6762	54.8524	48.1958	60.4762	71.3278	66.1119	77.1117	87.2646
		0.2	41.6668	53.5031	47.0501	58.9838	69.5314	64.4759	75.1590	85.0193
		0.3	40.6080	52.0832	45.8450	57.4147	67.6335	62.7463	73.1040	82.6455
III										
0	0.5	0.1	60.2275	73.5901	68.3596	81.6515	93.1589	89.7392	101.3089	111.8524
		0.2	58.655	71.5484	66.5823	79.4013	90.5064	87.276	98.4313	108.6038
		0.3	57.0011	69.3965	64.7029	77.0308	87.7033	84.6537	95.3998	105.1614
	5	0.1	74.8632	92.8234	85.1805	103.158	118.7498	113.487	129.2114	143.5751
		0.2	73.0156	90.391	83.0726	100.4543	115.5544	110.5087	125.7064	139.6032
		0.3	71.0805	87.8301	80.8484	97.6119	112.1653	107.3421	122.019	135.3932
10	0.5	0.1	68.8676	82.976	76.9429	90.768	100.0929	98.5898	107.9556	116.5201
		0.2	67.1165	80.8025	74.9819	88.3855	97.3856	95.993	105.0206	113.2514
		0.3	65.2706	78.5054	72.9039	85.8693	94.5196	93.2184	101.9241	109.7853
	5	0.1	84.2682	102.3502	94.6153	112.4965	125.4685	122.6266	135.7066	147.9869
		0.2	82.2198	99.7719	92.2958	109.6448	122.209	119.4972	132.1325	143.985
		0.3	80.074	97.0495	89.8369	106.6403	118.7445	116.1535	128.3674	139.7404
100	0.5	0.1	69.7166	87.6965	77.9318	95.9424	110.0154	104.1782	118.2545	129.4072
		0.2	67.935	85.3667	75.9353	93.3938	107.0535	101.4055	115.0592	125.8909
		0.3	66.0576	82.9043	73.8183	90.702	103.9124	98.438	111.6831	122.1494
	5	0.1	85.284	107.734	95.8027	118.4093	136.1596	129.0226	146.8399	161.2308
		0.2	83.1993	104.979	93.4406	115.3673	132.6257	125.6891	142.9752	156.9566
		0.3	81.0165	102.0692	90.9347	112.1619	128.8584	122.1173	138.8984	152.4091
10^{20}	0.5	0.1	69.8203	88.4832	78.0546	96.8516	112.8523	105.2108	121.4183	136.0367
		0.2	68.0348	86.1215	76.0535	94.2667	109.7776	102.397	118.0963	132.2632
		0.3	66.1534	83.626	73.9314	91.5371	106.5157	99.3792	114.5866	128.236
	5	0.1	85.4094	108.6728	95.9521	119.4974	139.4954	130.2621	150.5761	168.8815
		0.2	83.3201	105.8799	93.5845	116.4117	135.834	126.8787	146.5598	164.3089
		0.3	81.1324	102.9301	91.0727	113.1607	131.922	123.2459	142.3232	159.4308

Table 9.2
Values of frequency parameter Ω for K = 10.0

		α	-0.5		0.0			0.5		
K_ϕ	p	ζ \ β	0.0	0.5	-0.5	0.0	0.5	-0.5	0.0	0.5
						I				
0	0.5	0.1	6.1365	6.3633	6.9710	7.0905	7.5683	7.8146	8.2260	8.8482
		0.2	5.9258	6.1535	6.7266	6.8521	7.3321	7.5482	7.9640	8.5889
		0.3	5.7045	5.9337	6.4696	6.6026	7.0854	7.2681	7.6906	8.3188
	5	0.1	11.3841	14.2137	13.3747	15.9992	19.1964	17.8335	20.9105	24.2773
		0.2	11.1297	13.9171	13.0708	15.6583	18.8084	17.4470	20.4802	23.7968
		0.3	10.8662	13.6097	12.7557	15.3051	18.4061	17.0466	20.0345	23.2983
10	0.5	0.1	8.8054	11.1433	9.5898	11.8212	13.3259	12.4534	13.8070	14.6076
		0.2	8.5727	10.8947	9.3321	11.5588	13.0715	12.1759	13.5420	14.3585
		0.3	8.3277	10.6320	9.0606	11.2822	12.8018	11.8822	13.2628	14.0955
	5	0.1	13.9782	17.7396	15.9687	19.5795	22.5904	21.4053	24.2801	27.0514
		0.2	13.6794	17.3779	15.6240	19.1808	22.1501	20.9678	23.8054	26.5361
		0.3	13.3681	16.9996	15.2646	18.7643	21.6894	20.5109	23.3097	25.9979
100	0.5	0.1	8.9262	11.9216	9.7432	12.7815	15.7554	13.5943	16.5904	19.3346
		0.2	8.6898	11.6433	9.4815	12.4871	15.4171	13.2836	16.2412	18.9494
		0.3	8.441	11.3497	9.2056	12.1771	15.0593	12.9547	15.8732	18.5413
	5	0.1	14.1282	18.6236	16.1572	20.6532	24.976	22.66	26.9743	30.9832
		0.2	13.8245	18.2279	15.8067	20.2168	24.451	22.1821	26.4123	30.3442
		0.3	13.5079	17.8142	15.4414	19.761	23.9012	21.6827	25.8246	29.6742
10^{20}	0.5	0.1	8.9402	12.0239	9.7613	12.9129	16.1585	13.7569	17.0840	20.3996
		0.2	8.7034	11.7412	9.4991	12.6134	15.8019	13.4406	16.7140	19.9650
		0.3	8.4541	11.4431	9.2226	12.2981	15.4252	13.1054	16.3246	19.5060
	5	0.1	14.1460	18.7490	16.1798	20.8125	25.4574	22.8548	27.5563	32.2059
		0.2	13.8416	18.3479	15.8286	20.3697	24.9105	22.3695	26.9694	31.5104
		0.3	13.5244	17.9288	15.4626	19.9073	24.3382	21.8625	26.3559	30.7821
						II				
0	0.5	0.1	26.2107	31.0602	30.1170	34.8682	39.3164	38.7236	43.1747	47.4227
		0.2	25.5063	30.1808	29.3172	33.8937	38.1848	37.6528	41.9430	46.0403
		0.3	24.7674	29.2568	28.4743	32.8706	36.9943	36.5213	40.6504	44.5859
	5	0.1	35.7673	43.9119	41.2659	49.3129	56.8297	54.7617	62.2799	69.4649
		0.2	34.9071	42.8056	40.2734	48.0694	55.3636	53.3810	60.6686	67.6396
		0.3	34.0078	41.6473	39.2335	46.7676	53.8251	51.9308	58.9812	65.7238
10	0.5	0.1	31.8597	38.5590	35.5937	42.0281	45.8619	45.5614	49.3148	52.3154
		0.2	31.0517	37.5841	34.6911	40.9628	44.6912	44.4028	48.0444	50.9268
		0.3	30.2014	36.5539	33.7348	39.8387	43.4542	43.1713	46.7059	49.4617
	5	0.1	42.1007	51.4357	47.5120	56.6144	62.8795	61.8348	68.0538	73.7554
		0.2	41.1116	50.2092	46.3900	55.2552	61.3507	60.3415	66.3812	71.9011
		0.3	40.0739	48.9184	45.2104	53.8259	59.7396	58.7656	64.6237	69.9495

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100	0.5	0.1	32.2644	41.0305	36.073	44.8008	52.0103	48.6076	55.7818	61.6087
		0.2	31.4422	39.9693	35.1541	43.6443	50.6676	47.3534	54.3396	60.0252
		0.3	30.5771	38.8482	34.1804	42.424	49.2458	46.018	52.817	58.3463
	5	0.1	42.6145	54.3901	48.1215	59.93	69.633	65.4806	75.1856	83.2071
		0.2	41.6073	53.0598	46.9785	58.4595	67.907	63.8693	73.3116	81.1286
		0.3	40.5508	51.6597	45.7765	56.9131	66.0835	62.1667	71.3388	78.9317
10^{20}	0.5	0.1	32.3127	41.3969	36.1311	45.2334	53.3800	49.1072	57.3379	64.9850
		0.2	31.4888	40.3206	35.2102	44.0598	51.9805	47.8340	55.8331	63.2648
		0.3	30.6218	39.1838	34.2342	42.8218	50.4992	46.4775	54.2452	61.4420
	5	0.1	42.6764	54.8526	48.1961	60.4764	71.3286	66.1121	77.1122	87.2684
		0.2	41.6669	53.5033	47.0504	58.9841	69.5321	64.4761	75.1597	85.0235
		0.3	40.6081	52.0835	45.8455	57.4151	67.6344	62.7469	73.1048	82.6513
III										
0	0.5	0.1	60.228	73.5904	68.3606	81.6519	93.1595	89.7399	101.3095	111.8567
		0.2	58.6556	71.5487	66.5836	79.4018	90.5069	87.2768	98.4321	108.6088
		0.3	57.0014	69.3968	64.7048	77.0314	87.704	84.6556	95.4009	105.1686
	5	0.1	74.8628	92.824	85.1797	103.1585	118.7512	113.4874	129.2122	143.5817
		0.2	73.0151	90.3916	83.0718	100.4549	115.5556	110.5092	125.7074	139.6104
		0.3	71.0804	87.8308	80.8471	97.6126	112.1667	107.3432	122.0203	135.4026
10	0.5	0.1	68.868	82.9764	76.9437	90.7687	100.093	98.5907	107.9557	116.5146
		0.2	67.1169	80.803	74.9827	88.3863	97.3857	95.9943	105.0208	113.245
		0.3	65.2708	78.5059	72.9051	85.8703	94.5197	93.2216	101.9244	109.7763
	5	0.1	84.2681	102.3516	94.6151	112.498	125.469	122.628	135.707	147.9782
		0.2	82.2197	99.7734	92.2956	109.6465	122.2094	119.4989	132.1329	143.9755
		0.3	80.0741	97.0513	89.8365	106.6423	118.745	116.1572	128.3678	139.7286
100	0.5	0.1	69.7169	87.6963	77.9324	95.9421	110.013	104.1777	118.252	129.3943
		0.2	67.9353	85.3664	75.9359	93.3934	107.0513	101.4048	115.0558	125.8758
		0.3	66.0577	82.9041	73.8193	90.7014	103.9095	98.4361	111.6785	122.1277
	5	0.1	85.2839	107.7328	95.8025	118.408	136.1518	129.0215	146.8353	161.209
		0.2	83.1992	104.9777	93.4403	115.3658	132.6185	125.6878	142.9697	156.9331
		0.3	81.0166	102.0676	90.9342	112.1602	128.8499	122.1145	138.8915	152.38
10^{20}	0.5	0.1	69.8205	88.4835	78.055	96.8519	112.8538	105.2112	121.42	136.0482
		0.2	68.035	86.1218	76.054	94.2672	109.779	102.3976	118.0986	132.2769
		0.3	66.1535	83.6262	73.932	91.5377	106.5176	99.3812	114.5898	128.2574
	5	0.1	85.4091	108.6733	95.9517	119.4977	139.5007	130.2623	150.5792	168.9019
		0.2	83.3198	105.8805	93.5839	116.4122	135.8388	126.8791	146.5635	164.3315
		0.3	81.1324	102.9309	91.0717	113.1613	131.9278	123.247	142.3278	159.4607

Table 9.3
Values of frequency parameter Ω for K = 100.0

		α	-0.5		0.0			0.5		
K_ϕ	p	ζ \ β	0.0	0.5	-0.5	0.0	0.5	-0.5	0.0	0.5
					I					
0	0.5	0.1	6.1737	6.3657	7.0042	7.093	7.5691	7.8171	8.2267	8.8484
		0.2	5.9637	6.1559	6.7607	6.8547	7.3329	7.551	7.9648	8.5892
		0.3	5.746	5.9362	6.5028	6.6053	7.0862	7.272	7.6915	8.3191
	5	0.1	11.4044	14.2141	13.3903	15.9996	19.1963	17.8338	20.9104	24.2772
		0.2	11.1502	13.9175	13.0859	15.6586	18.8083	17.4472	20.4801	23.7966
		0.3	10.8904	13.6101	12.7676	15.3054	18.406	17.0464	20.0343	23.2981
10	0.5	0.1	8.8157	11.1441	9.6017	11.8223	13.3263	12.4547	13.8073	14.6078
		0.2	8.5829	10.8955	9.3444	11.56	13.0718	12.1773	13.5424	14.3587
		0.3	8.3378	10.6328	9.0735	11.2834	12.8022	11.8845	13.2633	14.0957
	5	0.1	13.9843	17.7396	15.9744	19.5796	22.5903	21.4055	24.28	27.0513
		0.2	13.6856	17.3779	15.6294	19.1808	22.15	20.9679	23.8053	26.536
		0.3	13.3756	16.9996	15.2683	18.7643	21.6893	20.5107	23.3096	25.9977
100	0.5	0.1	8.9317	11.9221	9.7499	12.7822	15.7557	13.5951	16.5907	19.3343
		0.2	8.6953	11.6438	9.4885	12.4878	15.4173	13.2845	16.2415	18.9491
		0.3	8.4461	11.3502	9.2136	12.1778	15.0596	12.9563	15.8736	18.5409
	5	0.1	14.1311	18.6235	16.1597	20.6532	24.9759	22.6601	26.9742	30.9829
		0.2	13.8274	18.2278	15.8091	20.2168	24.4509	22.1821	26.4122	30.344
		0.3	13.5117	17.8142	15.4426	19.761	23.9011	21.6825	25.8245	29.6739
10^{20}	0.5	0.1	8.9449	12.0242	9.767	12.9133	16.1587	13.7574	17.0842	20.4002
		0.2	8.708	11.7415	9.5051	12.6138	15.8021	13.4411	16.7143	19.9657
		0.3	8.4583	11.4434	9.2296	12.2986	15.4255	13.1066	16.3249	19.5069
	5	0.1	14.1482	18.749	16.1817	20.8125	25.4574	22.8548	27.5563	32.2061
		0.2	13.8438	18.3479	15.8304	20.3697	24.9105	22.3694	26.9693	31.5106
		0.3	13.5274	17.9287	15.4633	19.9073	24.3382	21.8623	26.3559	30.7823
					II					
0	0.5	0.1	26.2153	31.0601	30.1171	34.8678	39.3165	38.7232	43.175	47.4214
		0.2	25.5111	30.1806	29.316	33.8933	38.1849	37.6522	41.9433	46.0389
		0.3	24.7761	29.2566	28.4677	32.8702	36.9945	36.5197	40.6507	44.584
	5	0.1	35.7764	43.9128	41.2759	49.3139	56.8308	54.7626	62.2806	69.4641
		0.2	34.9162	42.8065	40.2841	48.0705	55.3645	53.3822	60.6695	67.6387
		0.3	34.016	41.6483	39.2457	46.7689	53.8262	51.9331	58.9824	65.7227
10	0.5	0.1	31.8605	38.5586	35.5931	42.0277	45.8615	45.5611	49.3145	52.3149
		0.2	31.0526	37.5837	34.6896	40.9624	44.6909	44.4024	48.044	50.9263
		0.3	30.2042	36.5536	33.7296	39.8382	43.4538	43.1699	46.7055	49.461
	5	0.1	42.1051	51.4364	47.5183	56.6153	62.8801	61.8357	68.0541	73.7555
		0.2	41.116	50.21	46.3967	55.2562	61.3511	60.3427	66.3816	71.9011
		0.3	40.0773	48.9192	45.2181	53.827	59.7401	58.7677	64.6242	69.9496

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100	0.5	0.1	32.2644	41.0303	36.0719	44.8006	52.0103	48.6074	55.7819	61.6065
		0.2	31.4423	39.969	35.1525	43.6441	50.6676	47.3532	54.3397	60.0227
		0.3	30.5784	38.848	34.1763	42.4238	49.2459	46.0174	52.8173	58.3429
	5	0.1	42.6171	54.3906	48.1254	59.9307	69.6339	65.4814	75.1862	83.2049
		0.2	41.6098	53.0604	46.9826	58.4603	67.9078	63.8702	73.3124	81.1262
		0.3	40.5526	51.6604	45.7816	56.9139	66.0845	62.1685	71.3398	78.9285
10^{20}	0.5	0.1	32.3126	41.3968	36.13	45.2333	53.3805	49.1072	57.3384	64.9881
		0.2	31.4887	40.3204	35.2086	44.0598	51.981	47.8339	55.8337	63.2683
		0.3	30.6229	39.1837	34.2305	42.8217	50.4997	46.4775	54.246	61.4467
	5	0.1	42.6786	54.8531	48.1995	60.4769	71.3299	66.1126	77.113	87.2724
		0.2	41.6691	53.5038	47.0541	58.9847	69.5332	64.4768	75.1607	85.028
		0.3	40.6096	52.084	45.8502	57.4158	67.6357	62.7484	73.1061	82.6569
III										
0	0.5	0.1	60.236	73.5916	68.372	81.6537	93.1618	89.7421	101.3119	111.8529
		0.2	58.6635	71.5499	66.5962	79.4038	90.509	87.2797	98.4352	108.6044
		0.3	57.0072	69.3981	64.7215	77.0337	87.7066	84.6619	95.405	105.1623
	5	0.1	74.8604	92.8244	85.1737	103.1588	118.7559	113.4875	129.2151	143.5744
		0.2	73.013	90.3921	83.0644	100.4553	115.56	110.5095	125.711	139.6025
		0.3	71.0821	87.8315	80.835	97.6132	112.172	107.3438	122.0247	135.3927
10	0.5	0.1	68.8722	82.9772	76.9508	90.7702	100.0937	98.5926	107.9563	116.5139
		0.2	67.121	80.8038	74.9904	88.3879	97.3863	95.9967	105.0216	113.2443
		0.3	65.2736	78.5068	72.9151	85.8721	94.5204	93.2267	101.9254	109.7751
	5	0.1	84.2661	102.3518	94.6115	112.4981	125.4693	122.628	135.707	147.9759
		0.2	82.2177	99.7738	92.2908	109.6467	122.2095	119.499	132.133	143.9731
		0.3	80.0748	97.0518	89.8273	106.6426	118.7452	116.157	128.3681	139.7255
100	0.5	0.1	69.7196	87.6969	77.9373	95.9432	110.0147	104.1792	118.2537	129.3864
		0.2	67.9379	85.3671	75.9415	93.3947	107.0527	101.4069	115.0581	125.8665
		0.3	66.0593	82.9048	73.8273	90.703	103.9114	98.4411	111.6816	122.1138
	5	0.1	85.283	107.7335	95.8008	118.4088	136.156	129.0222	146.8378	161.1948
		0.2	83.1984	104.9786	93.4382	115.3668	132.6224	125.6886	142.9727	156.9177
		0.3	81.0172	102.0687	90.9302	112.1613	128.8545	122.1163	138.8952	152.3604
10^{20}	0.5	0.1	69.8229	88.484	78.0596	96.8529	112.856	105.2126	121.4222	136.0633
		0.2	68.0374	86.1224	76.0591	94.2684	109.781	102.3995	118.1015	132.2945
		0.3	66.1549	83.6269	73.9397	91.5391	106.5201	99.3863	114.5939	128.2828
	5	0.1	85.4083	108.6745	95.9501	119.4991	139.5076	130.2635	150.583	168.9267
		0.2	83.319	105.8819	93.582	116.4138	135.8449	126.8807	146.5683	164.358
		0.3	81.1329	102.9325	91.0682	113.1633	131.9351	123.251	142.3339	159.4932

Table 9.4
Values of frequency parameter Ω for K = 10^{20}

		α	-0.5		0.0			0.5		
K_ϕ	p	ζ \ β	0.0	0.5	-0.5	0.0	0.5	-0.5	0.0	0.5
					I					
0	0.5	0.1	3.0511	3.6063	3.4313	3.9591	4.4845	4.3072	4.8126	5.3627
		0.2	2.9580	3.4959	3.3244	3.8356	4.3528	4.1709	4.6682	5.2138
		0.3	2.8601	3.3803	3.2112	3.7062	4.2153	4.0269	4.5176	5.0587
	5	0.1	5.1926	7.6959	5.9234	8.4054	10.9841	9.1254	11.6728	14.3152
		0.2	5.0975	7.5623	5.8142	8.2563	10.8004	8.9610	11.4733	14.0821
		0.3	4.9992	7.4244	5.7013	8.1024	10.6109	8.7913	11.2675	13.8418
10	0.5	0.1	5.3676	7.6200	5.6893	7.8328	9.0126	8.0235	9.0942	9.6417
		0.2	5.2513	7.4848	5.5635	7.6944	8.8808	7.8803	8.9583	9.5137
		0.3	5.1285	7.3415	5.4298	7.5482	8.7411	7.7277	8.8153	9.3790
	5	0.1	7.8147	11.3077	8.5726	11.9797	14.3241	12.6242	14.8937	16.9550
		0.2	7.6843	11.1315	8.4297	11.7945	14.1157	12.4289	14.6750	16.7110
		0.3	7.5483	10.9469	8.2808	11.6009	13.8980	12.2250	14.4472	16.4572
100	0.5	0.1	5.5184	8.6004	5.8742	8.985	11.8276	9.3336	12.1732	14.6247
		0.2	5.3974	8.4325	5.7432	8.8131	11.6191	9.1569	11.9643	14.3932
		0.3	5.2698	8.255	5.6038	8.6317	11.3979	8.9686	11.7437	14.1472
	5	0.1	8.0348	12.6219	8.8475	13.5233	17.6322	14.3773	18.4938	22.0191
		0.2	7.898	12.4048	8.6972	13.2949	17.3371	14.1372	18.1904	21.6681
		0.3	7.7554	12.1776	8.5406	13.0561	17.0277	13.8862	17.8727	21.2995
10^{20}	0.5	0.1	5.5364	8.7410	5.8966	9.1590	12.3899	9.5419	12.8387	16.1019
		0.2	5.4148	8.5676	5.7649	8.9809	12.1586	9.3584	12.6051	15.8115
		0.3	5.2866	8.3843	5.6246	8.7930	11.9139	9.1624	12.3589	15.5040
	5	0.1	8.0616	12.8290	8.8817	13.7813	18.4492	14.6875	19.4595	24.0902
		0.2	7.9240	12.6041	8.7304	13.5439	18.1226	14.4373	19.1214	23.6603
		0.3	7.7805	12.3690	8.5728	13.2960	17.7805	14.1759	18.7678	23.2097
					II					
0	0.5	0.1	19.2579	24.3030	21.5818	26.8648	31.3766	29.3989	34.0766	38.3525
		0.2	18.7611	23.6512	21.0378	26.1600	30.5293	28.6425	33.1710	37.3076
		0.3	18.2392	22.9653	20.4605	25.4189	29.6363	27.8366	32.2192	36.2055
	5	0.1	27.0025	34.4671	30.7198	38.4355	45.1860	42.3682	49.2944	55.7075
		0.2	26.3778	33.6269	30.0134	37.5045	44.0558	41.3488	48.0672	54.2861
		0.3	25.7246	32.7469	29.2727	36.5298	42.8698	40.2769	46.7815	52.7934
10	0.5	0.1	24.2921	29.9494	26.5502	32.2123	35.5334	34.4941	37.9649	41.1398
		0.2	23.7009	29.2326	25.9108	31.4432	34.6637	33.6736	37.0343	40.0880
		0.3	23.0778	28.4744	25.2291	30.6308	33.7440	32.7942	36.0536	38.9768
	5	0.1	32.9268	40.4837	36.6065	44.1842	49.3148	47.8861	53.1977	58.3770
		0.2	32.1856	39.5651	35.7812	43.1753	48.1527	46.7887	51.9365	56.9415
		0.3	31.4078	38.5984	34.9120	42.1145	46.9295	45.6294	50.6120	55.4319

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100	0.5	0.1	24.8984	33.1984	27.2635	35.745	42.1402	38.2863	44.7293	49.5179
		0.2	24.2864	32.381	26.6	34.8711	41.1182	37.3564	43.6461	48.3278
		0.3	23.6416	31.5163	25.8909	33.9477	40.0339	36.3581	42.5008	47.064
	5	0.1	33.7359	44.5476	37.5575	48.5981	56.9301	52.6185	61.0107	67.4023
		0.2	32.9674	43.5062	36.7006	47.4598	55.5916	51.3852	59.57	65.8084
		0.3	32.1611	42.4097	35.7977	46.2622	54.1774	50.0797	58.0524	64.1242
10^{20}	0.5	0.1	24.9742	33.7775	27.3546	36.4180	44.2366	39.0554	47.0680	54.3891
		0.2	24.3595	32.9373	26.6877	35.5183	43.1339	38.0966	45.8967	53.0218
		0.3	23.7119	32.0488	25.9733	34.5680	41.9647	37.0635	44.6590	51.5693
	5	0.1	33.8386	45.3252	37.6812	49.5021	59.6900	53.6513	64.0875	73.6201
		0.2	33.0665	44.2538	36.8199	48.3297	58.2475	52.3796	62.5325	71.8038
		0.3	32.2564	43.1262	35.9119	47.0965	56.7238	51.0319	60.8951	69.8832
III										
0	0.5	0.1	49.7268	62.594	55.7345	68.8595	79.8955	75.0904	86.3419	96.4467
		0.2	48.4515	60.9046	54.309	67.0138	77.6922	73.0851	83.9656	93.7351
		0.3	47.1099	59.1221	52.7898	65.0672	75.3596	70.9291	81.4587	90.8538
	5	0.1	62.6795	79.4445	70.6795	87.7908	102.2538	96.0734	110.8234	124.11
		0.2	61.1588	77.4044	68.9584	85.5372	99.5581	93.6049	107.8817	120.7436
		0.3	59.5656	75.2558	67.1352	83.1664	96.6978	90.9659	104.7837	117.1693
10	0.5	0.1	57.4046	69.2153	63.4308	75.2684	83.9895	81.3054	90.2789	98.9728
		0.2	55.9714	67.4417	61.8413	73.3363	81.7538	79.2117	87.8678	96.2463
		0.3	54.4608	65.5664	60.142	71.295	79.3847	76.9525	85.3219	93.348
	5	0.1	71.1851	86.2759	79.2622	94.445	106.3143	102.5637	114.753	126.5711
		0.2	69.4882	84.145	77.3559	92.0977	103.5832	99.9985	111.7719	123.1868
		0.3	67.7086	81.8961	75.3287	89.6242	100.6813	97.245	108.63	119.5918
100	0.5	0.1	58.7453	75.2332	64.9793	81.7069	93.8466	88.1264	100.3322	109.5201
		0.2	57.2651	73.2827	63.3356	79.5888	91.3962	85.837	97.7007	106.6256
		0.3	55.7057	71.2196	61.5729	77.3497	88.7939	83.3597	94.9168	103.5406
	5	0.1	72.8242	93.2263	81.1572	101.8889	117.0633	110.4559	125.7435	137.5798
		0.2	71.071	90.8959	79.1853	99.3294	114.1039	107.6658	122.5197	134.0148
		0.3	69.2334	88.4344	77.0837	96.6308	110.9474	104.6585	119.1158	130.2172
10^{20}	0.5	0.1	58.9204	76.5395	65.1863	83.2009	98.2801	89.8106	105.2082	118.9864
		0.2	57.4336	74.5377	63.5343	81.0249	95.664	87.4561	102.3951	115.7621
		0.3	55.8676	72.4211	61.7579	78.7254	92.8861	84.8939	99.4203	112.3172
	5	0.1	73.0413	94.8352	81.4146	103.7316	122.4367	112.5353	131.6569	148.7729
		0.2	71.2801	92.4429	79.4328	101.1016	119.2844	109.6653	128.2149	144.8226
		0.3	69.4346	89.9168	77.3169	98.3293	115.918	106.5564	124.581	140.6026

Table 9.5

Values of frequency parameter Ω for clamped plate for v_0=0.3

		α							
		-0.5		0.0			0.5		
		β		β			β		
p	ζ	0.0	0.5	-0.5	0.0	0.5	-0.5	0.0	0.5
					I				
0.5	0	5.6525	8.9061	6.0223	9.3288	12.6105	9.7155	13.0603	16.3793
	0.1	5.5364	8.7410	5.8966	9.1590	12.3899	9.5419	12.8387	16.1019
	0.2	5.4148	8.5676	5.7649	8.9809	12.1586	9.3584	12.6051	15.8115
	0.3	5.2866	8.3843	5.6246	8.7930	11.9139	9.1624	12.3589	15.5040
1	0	6.1504	9.6733	6.6320	10.2158	13.7310	10.7241	14.3022	17.8332
	0.1	6.0312	9.4979	6.5019	10.0331	13.4909	10.5336	14.0571	17.5281
	0.2	5.9065	9.3141	6.3658	9.8418	13.2392	10.3344	13.8005	17.2084
	0.3	5.7751	9.1204	6.2227	9.6406	12.9741	10.1252	13.5306	16.8716
2	0	6.8448	10.7879	7.4619	11.4865	15.3837	12.1492	16.1193	20.0109
	0.1	6.7212	10.5986	7.3260	11.2875	15.1180	11.9400	15.8456	19.6679
	0.2	6.5922	10.4008	7.1843	11.0796	14.8402	11.7216	15.5594	19.3094
	0.3	6.4570	10.1930	7.0356	10.8615	14.5482	11.4924	15.2590	18.9323
	.				II				
0.5	0	25.5616	34.5777	27.9912	37.2743	45.2878	39.9582	48.1751	55.6945
	0.1	24.9742	33.7775	27.3546	36.4180	44.2366	39.0554	47.0680	54.3891
	0.2	24.3595	32.9373	26.6877	35.5183	43.1339	38.0966	45.8967	53.0218
	0.3	23.7119	32.0488	25.9733	34.5680	41.9647	37.0635	44.6590	51.5693
1	0	27.3004	36.7864	30.0152	39.7711	48.1833	42.7395	51.3487	59.2193
	0.1	26.6825	35.9411	29.3411	38.8605	47.0645	41.7657	50.1602	57.8314
	0.2	26.0341	35.0532	28.6339	37.9044	45.8894	40.7435	48.9120	56.3737
	0.3	25.3498	34.1155	27.8880	36.8951	44.6483	39.6643	47.5940	54.8336
2	0	29.7718	39.9938	32.9008	43.4139	52.4724	46.8084	56.0813	64.5556
	0.1	29.1102	39.0868	32.1720	42.4302	51.2644	45.7527	54.7925	63.0474
	0.2	28.4186	38.1359	31.4092	41.3984	49.9979	44.6420	53.4371	61.4658
	0.3	27.6914	37.1327	30.6024	40.3106	48.6588	43.4659	52.0074	59.7945
					III				
0.5	0	60.3410	78.4457	66.7627	85.2709	100.7752	92.0171	107.8569	122.0720
	0.1	58.9204	76.5395	65.1863	83.2009	98.2801	89.8106	105.2082	118.9864
	0.2	57.4336	74.5377	63.5343	81.0249	95.6640	87.4561	102.3951	115.7621
	0.3	55.8676	72.4211	61.7579	78.7254	92.8861	84.8939	99.4203	112.3172
1	0	63.0618	81.9009	69.8624	89.1041	105.2133	96.2274	112.6399	127.3516
	0.1	61.5896	79.9173	68.2289	86.9445	102.6045	93.8954	109.8486	124.1400
	0.2	60.0435	77.8331	66.5131	84.6753	99.8627	91.4436	106.9132	120.7663
	0.3	58.4104	75.6305	64.7006	82.2772	96.9643	88.8478	103.8101	117.1934
2	0	67.0271	87.0376	74.4187	94.8696	111.9901	102.5974	120.0520	135.6973
	0.1	65.4745	84.9482	72.6895	92.5887	109.2373	100.1382	117.1009	132.2879
	0.2	63.8536	82.7576	70.8782	90.1926	106.3503	97.5399	113.9879	128.7089
	0.3	62.1496	80.4419	68.9498	87.6624	103.2864	94.7736	110.6982	124.9208

Table 9.6

Values of frequency parameter Ω for simply supported plate for ν_0=0.3

		α							
		-0.5		0.0			0.5		
		β		β			β		
p	ζ	0.0	0.5	-0.5	0.0	0.5	-0.5	0.0	0.5
					I				
0.5	0	3.1402	3.7122	3.5335	4.0775	4.6113	4.437	4.951	5.5068
	0.1	3.0511	3.6063	3.4313	3.9591	4.4845	4.3072	4.8126	5.3627
	0.2	2.9580	3.4959	3.3244	3.8356	4.3528	4.1709	4.6682	5.2138
	0.3	2.8601	3.3803	3.2112	3.7062	4.2153	4.0269	4.5176	5.0587
1	0	3.5498	4.4716	4.0392	4.9351	5.8537	5.4003	6.2928	7.2492
	0.1	3.4610	4.3624	3.9369	4.8122	5.7181	5.2639	6.1435	7.0902
	0.2	3.3684	4.2492	3.8303	4.6847	5.5778	5.1224	5.9890	6.9262
	0.3	3.2716	4.1313	3.7188	4.5518	5.4323	4.9750	5.8287	6.7564
2	0	4.1291	5.5757	4.7239	6.1461	7.6237	6.7238	8.1704	9.6986
	0.1	4.0396	5.4606	4.6208	6.0163	7.4744	6.5796	8.0062	9.5172
	0.2	3.9469	5.3417	4.5138	5.8821	7.3206	6.4304	7.8368	9.3304
	0.3	3.8505	5.2185	4.4024	5.7430	7.1615	6.2756	7.6616	9.1377
					II				
0.5	0	19.7337	24.9263	22.1023	27.5384	32.188	30.1156	34.9378	39.3547
	0.1	19.2579	24.3030	21.5818	26.8648	31.3766	29.3989	34.0766	38.3525
	0.2	18.7611	23.6512	21.0378	26.1600	30.5293	28.6425	33.1710	37.3076
	0.3	18.2392	22.9653	20.4605	25.4189	29.6363	27.8366	32.2192	36.2055
1	0	21.2387	26.8325	23.887	29.72	34.729	32.5693	37.7427	42.4887
	0.1	20.7379	26.1725	23.3342	29.0012	33.8619	31.7942	36.8143	41.4165
	0.2	20.2140	25.4824	22.7560	28.2496	32.9556	30.9837	35.8440	40.2962
	0.3	19.6631	24.7571	22.1480	27.4598	32.0036	30.1319	34.8248	39.1194
2	0	23.3828	29.6222	26.4284	32.9117	38.5145	36.1592	41.9307	47.2429
	0.1	22.8468	28.9120	25.8303	32.1322	37.5729	35.3142	40.9175	46.0702
	0.2	22.2879	28.1706	25.2059	31.3180	36.5902	34.4296	39.8576	44.8462
	0.3	21.7020	27.3923	24.5490	30.4636	35.5574	33.4986	38.7453	43.5605
					III				
0.5	0	50.9465	64.2057	57.0959	70.6186	82.0004	76.9790	88.5882	99.0439
	0.1	49.7268	62.5940	55.7345	68.8595	79.8955	75.0904	86.3419	96.4467
	0.2	48.4515	60.9046	54.3090	67.0138	77.6922	73.0851	83.9656	93.7351
	0.3	47.1099	59.1221	52.7898	65.0672	75.3596	70.9291	81.4587	90.8538
1	0	53.4409	67.3817	59.9533	74.1561	86.1120	80.8740	93.0367	103.9816
	0.1	52.1745	65.7009	58.5381	72.3164	83.9063	78.8776	90.6666	101.2683
	0.2	50.8464	63.9384	57.0538	70.3870	81.5929	76.7828	88.1797	98.4235
	0.3	49.4460	62.0796	55.4881	68.3521	79.1531	74.5711	85.5567	95.4206
2	0	57.0633	72.0834	64.1360	79.4452	92.3447	86.7342	99.8640	111.6876
	0.1	55.7265	70.3083	62.6368	77.4967	90.0080	84.6206	97.3490	108.7967
	0.2	54.3315	68.4501	61.0676	75.4537	87.5620	82.3945	94.7032	105.7688
	0.3	52.8662	66.4906	59.4059	73.3007	84.9748	80.0354	91.9137	102.5719

Table 9.7
Values of frequency parameter Ω for free plate for ν_θ=0.3

		α							
		-0.5		0.0			0.5		
		β		β			β		
p	ζ	0.0	0.5	-0.5	0.0	0.5	-0.5	0.0	0.5
					I				
0.5	0	6.3375	6.5646	7.2040	7.3191	7.7956	8.0688	8.4772	9.0986
	0.1	6.1357	6.3632	6.9699	7.0904	7.5682	7.8144	8.2259	8.8480
	0.2	5.9250	6.1534	6.7254	6.8520	7.3321	7.5480	7.9639	8.5887
	0.3	5.7039	5.9336	6.4680	6.6024	7.0853	7.2677	7.6904	8.3186
1	0	7.3210	8.0104	8.4538	9.0031	10.0019	10.0094	10.9166	12.0855
	0.1	7.1121	7.7961	8.2086	8.7570	9.7505	9.7316	10.6357	11.7991
	0.2	6.8946	7.5736	7.9533	8.5015	9.4902	9.4432	10.3448	11.5032
	0.3	6.6672	7.3418	7.6863	8.2353	9.2198	9.1428	10.0426	11.1964
2	0	8.7395	10.1358	10.2054	11.4280	13.1807	12.7519	14.3912	16.3278
	0.1	8.5192	9.8999	9.9440	11.1555	12.8912	12.4424	14.0671	15.9853
	0.2	8.2907	9.6558	9.6729	10.8734	12.5921	12.1219	13.7320	15.6317
	0.3	8.0530	9.4023	9.3905	10.5805	12.2818	11.7890	13.3847	15.2655
					II				
0.5	0	26.8863	31.9022	30.8834	35.8008	40.4010	39.7421	44.3485	48.7486
	0.1	26.2110	31.0602	30.1177	34.8682	39.3163	38.7235	43.1746	47.4216
	0.2	25.5066	30.1808	29.3180	33.8937	38.1847	37.6527	41.9429	46.0391
	0.3	24.7674	29.2568	28.4755	32.8706	36.9941	36.5211	40.6502	44.5843
1	0	28.6949	34.2396	33.0007	38.4432	43.5210	42.6847	47.7600	52.5990
	0.1	27.9892	33.3536	32.1967	37.4573	42.3710	41.5994	46.5081	51.1900
	0.2	27.2519	32.4281	31.3565	36.4277	41.1701	40.4658	45.2007	49.7190
	0.3	26.4775	31.4568	30.4743	35.3470	39.9100	39.2759	43.8290	48.1754
2	0	31.3203	37.7309	36.0782	42.3897	48.2698	47.0838	52.9632	58.5702
	0.1	30.5718	36.7846	35.2210	41.3320	47.0318	45.9167	51.6118	57.0433
	0.2	29.7918	35.7977	34.3265	40.2281	45.7407	44.6962	50.1992	55.4508
	0.3	28.9749	34.7625	33.3874	39.0707	44.3851	43.4142	48.7179	53.7797
					III				
0.5	0	61.7314	75.5397	70.0565	83.7983	95.6952	92.0679	104.0347	114.9669
	0.1	60.2275	73.5901	68.3596	81.6515	93.1589	89.7392	101.3089	111.8524
	0.2	58.6550	71.5484	66.5823	79.4013	90.5064	87.2760	98.4313	108.6038
	0.3	57.0011	69.3965	64.7029	77.0308	87.7033	84.6537	95.3998	105.1614
1	0	64.5155	79.1067	73.2292	87.7502	100.3099	96.3948	109.0018	120.4950
	0.1	62.9583	77.0796	71.4698	85.5135	97.6619	93.9498	106.1394	117.2496
	0.2	61.3265	74.9558	69.6255	83.1696	94.8873	91.3862	103.1385	113.8505
	0.3	59.6071	72.7183	67.6815	80.6998	91.9639	88.6827	99.9762	110.2659
2	0	68.6027	84.4560	77.9329	93.7427	107.4151	103.0112	116.7510	129.2827
	0.1	66.9669	82.3237	76.0812	91.3849	104.6209	100.4347	113.7263	125.8404
	0.2	65.2600	80.0932	74.1429	88.9145	101.6982	97.7254	110.5485	122.2368
	0.3	63.4675	77.7427	72.0976	86.3131	98.6104	94.8632	107.2009	118.4369

Table 9.8
Comparison of frequency parameter Ω for uniform isotropic circular plate

K	0				∞			
Mode \ K_ϕ	0	10	100	∞	0	10	100	∞
I	9.0030* 9.0031° 9.0031	13.5130* 13.5129° 13.5129	14.5390* 14.5388° 14.5388	14.6820* 14.6820° 14.6820	4.9350* 4.9351° 4.9351	8.7520* 9.7519° 8.7519	10.0190* 10.0192° 10.0192	10.2160* 10.2158° 10.2158
II	38.4430* 38.4432° 38.4432	45.7640* 45.7643° 45.7643	48.7460* 48.7457° 48.7457	49.2180* 49.2159° 49.2185	29.7200* 29.7200° 29.7200	35.2180* 35.2190° 35.2190	39.0290* 39.0288° 39.0288	39.7710* 39.7711° 39.7711
III	87.7490* 87.7502° 87.7502	97.0450* 97.0428° 97.0428	102.5180* 102.5204° 102.5204	103.5000* 103.4995° 103.4995	74.1560* 74.1560° 74.1561	80.6850* 80.6869° 80.6870	87.4880* 87.4900° 87.4901	89.1030* 89.1041° 89.1041

* values taken from Azimi[1988].
° values taken from Ansari[2000].

Table 9.9
Comparison of frequency parameter Ω for uniform isotropic circular plate

Method	Clamped			S-S		
F.E.M.	10.2159*	39.7766*	89.1708*	4.9352*	29.7222*	74.1938*
Receptence	10.2160°	39.7710°	89.1030°	4.9350°	29.7200°	74.1560°
Ritz	10.2158†	39.7711†	89.1041†	4.9351†	29.7200†	74.1560†
Exact	10.2158‡	39.7711‡	89.1041‡	4.9352‡	29.7200‡	74.1561‡
present	10.2158	39.7711	89.1041	4.9351	29.7200	74.1561

* values taken from Pardoen[1978].
° values taken from Azimi[1988].
† values taken from Ansari[2000].
‡ exact values taken from Leissa[1969].

Table 9.10

Comparison of frequency parameter Ω for parabolically tapered ($\alpha = 0$) orthotropic circular plate

β \ p^2	0.5	0.75	1	2	5
			Clamped		
-0.5	6.0167*	6.3543*	6.6320*	7.4614*	9.0266*
	6.0223	6.3549	6.6320	7.4619	9.0273
-0.3	7.3394*	7.7414*	8.0759*	9.0923*	11.0598*
	7.3468	7.7422	8.0759	9.0932	11.0601
-0.1	8.6602*	9.1195*	9.5055*	10.6932*	13.0263*
	8.6690	9.1206	9.5055	10.6944	13.0367
0	9.3194*	9.8057*	10.2158*	11.4852*	14.0090*
	9.3288	9.8068	10.2158	11.4865	14.0094
0.1	9.9775*	10.4898*	10.9235*	12.2723*	14.9731*
	9.9877	10.4911	10.9235	12.2739	14.9737
0.3	11.2905*	11.8526*	12.3317*	11.8342*	16.8795*
	11.3015	11.8540	12.3317	13.8360	16.8803
0.5	12.5988*	13.2086*	11.7310*	15.3818*	18.7620*
	12.6105	13.2100	13.7310	15.3837	18.7626
			S-S		
-0.5	3.5298*	3.8098*	4.0392*	4.7237*	6.0293*
	3.5335	3.8102	4.0392	4.7239	6.0293
-0.3	3.7562*	4.1101*	4.4034*	5.2936*	7.0320*
	3.7607	4.1106	4.4034	5.2940	7.0321
-0.1	3.9680*	4.3982*	4.7576*	5.3598*	8.0405*
	3.9730	4.3987	4.7576	5.8602	8.0406
0	4.0723*	4.5418*	4.9351*	6.1456*	8.5501*
	4.0775	4.5423	4.9351	6.1461	8.5503
0.1	4.1767*	4.6863*	5.1142*	6.4343*	9.0640*
	4.1822	4.6869	5.1142	6.4348	9.0642
0.3	4.3882*	4.9802*	5.4787*	7.0216*	10.1049*
	4.3938	4.9808	5.4787	7.0222	10.1050
0.5	4.6056*	5.2827*	5.8537*	7.6231*	11.1625*
	4.6113	5.2833	5.8537	7.6237	11.1627

* values taken from Ansari[2000].

Table 9.11

Comparison of frequency parameter Ω for linearly tapered (β = 0) and parabolically tapered (α = 0) orthotropic (p^2 = 1.44) circular plate subjected to thermal gradient

Mode	ζ	α=0, β=0		α=-0.3, β=0		α=0, β=-0.3	
		(Frobenius)*	present	Frobenius	present	Frobenius	present
Clamped							
I	0	10.8285	10.8292	8.2281	8.2287	8.5696	8.5701
	0.1	10.6377	10.6384	8.0785	8.0790	8.4137	8.4142
	0.2	10.4382	10.4389	7.9222	7.9227	8.2509	8.2513
	0.3	10.2286	10.2293	7.7580	7.7585	8.0800	8.0804
	0.4	10.0073	10.0080	7.5845	7.5850	7.8997	7.9001
II	0	41.5200	41.5281	33.8840	33.8906	35.7066	35.7124
	0.1	40.5736	40.5815	33.1191	33.1249	34.9025	34.9080
	0.2	39.5806	39.5882	32.3167	32.3225	34.0590	34.0641
	0.3	38.5331	38.5403	31.4704	31.4766	33.1697	33.1745
	0.4	37.4204	37.4277	30.5719	30.5778	32.2259	32.2312
III	0	91.8433	91.8749	76.1952	76.2207	80.5468	80.5690
	0.1	89.6248	89.6557	74.3899	74.4146	78.6378	78.6601
	0.2	87.2942	87.3240	72.4994	72.5212	76.6349	76.6549
	0.3	84.8319	84.8607	70.4979	70.5233	74.5191	74.5374
	0.4	82.2125	82.2406	68.3708	68.3953	72.2695	72.2888
S-S							
I	0	5.5202	5.5205	4.5140	4.5142	4.8356	4.8358
	0.1	5.3940	5.3943	4.4087	4.4089	4.7220	4.7223
	0.2	5.2634	5.2637	4.2995	4.2997	4.6042	4.6044
	0.3	5.1277	5.1279	4.1858	4.1860	4.4814	4.4816
	0.4	4.9861	4.9864	4.0669	4.0671	4.3531	4.3533
II	0	31.2503	31.2558	25.9639	25.9681	27.7302	27.7340
	0.1	30.5023	30.5072	25.3520	25.3558	27.0825	27.0862
	0.2	29.7201	29.7249	24.7125	24.7162	26.4055	26.4089
	0.3	28.8988	28.9033	24.0407	24.0446	25.6945	25.6977
	0.4	28.0310	28.0355	23.3308	23.3345	24.9432	24.9467
III	0	76.6757	76.6986	64.1305	64.1506	68.2644	68.2816
	0.1	74.7825	74.8054	62.5873	62.6039	66.6232	66.6404
	0.2	72.7980	72.8201	60.9680	60.9847	64.9043	64.9193
	0.3	70.7063	70.7271	59.2608	59.2789	63.0906	63.1051
	0.4	68.4854	68.5057	57.4487	57.4663	61.1654	61.1817

* values taken from Gupta[1984].

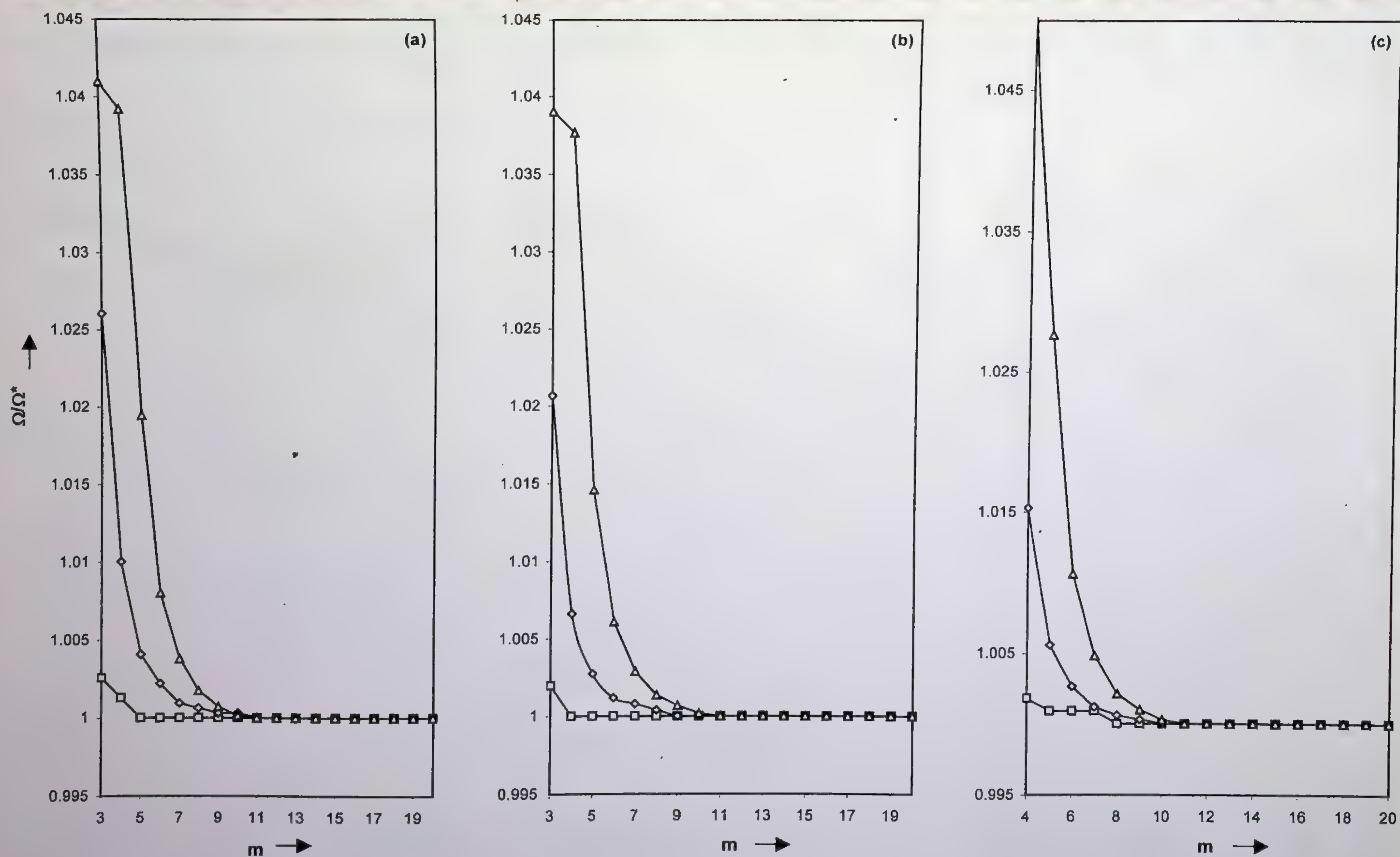

Fig.9.1 : Convergence of the Normalized Frequency Parameter Ω/Ω^* with no. of terms *m* used for the first three modes of vibration for $p = 5.0$, $\zeta = 0.5$, $\alpha = -0.3$, $\beta = -0.2$ for (a) Clamped (b) Simply supported and (c) Free plate.
——□——, Fundamental mode; ——◊——, Second mode; ——Δ——, Third mode. Ω^*- the frequency using 20 terms.

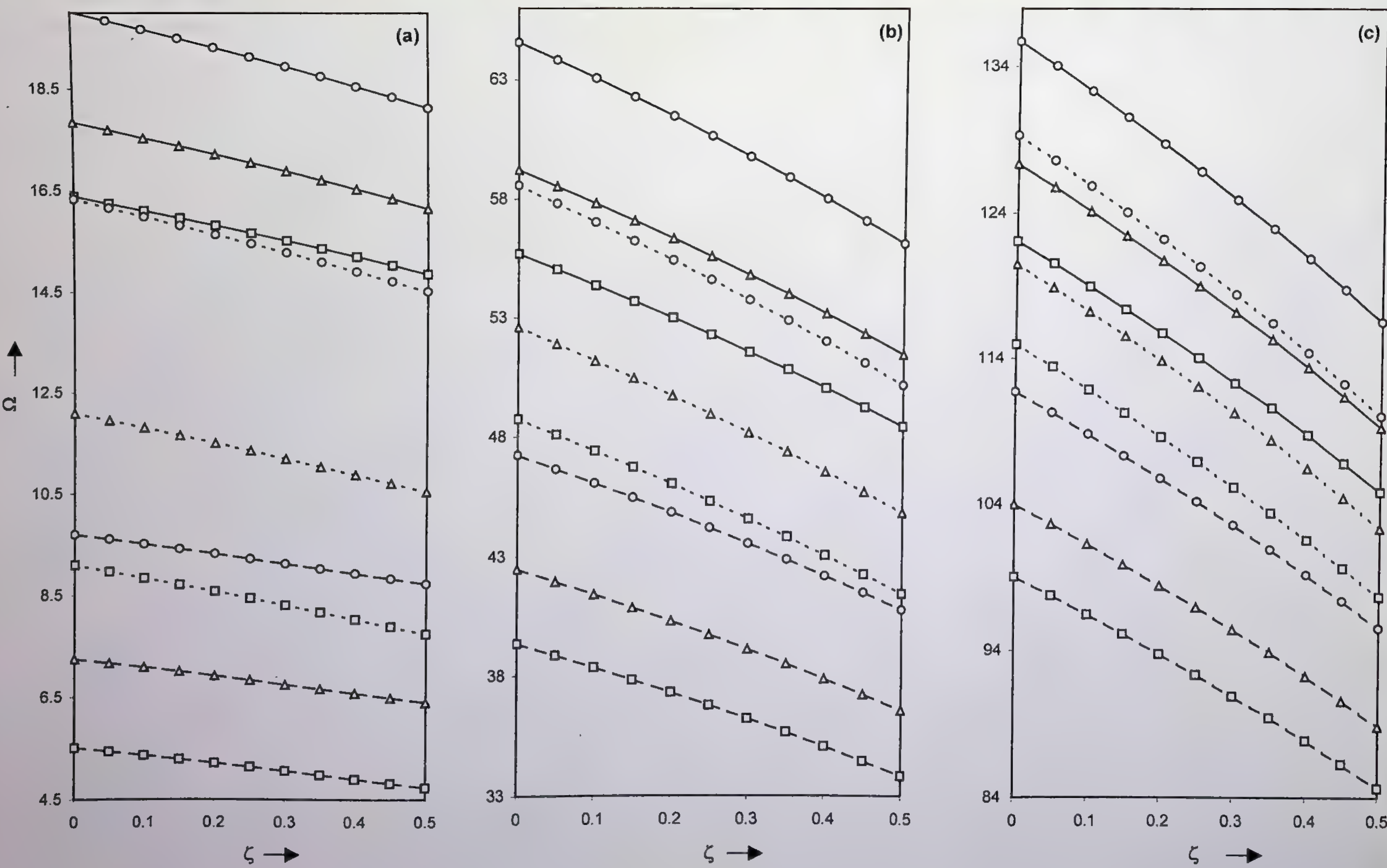

Fig. 9.2 : Frequency parameter of plates vibrating in (a) fundamental (b) second and (c) third mode for $\alpha = 0.5$, $\beta = 0.5$.
———, clamped; – – – – –, simply supported; -----------, free. □, $p = 0.5$; Δ, $p = 1.0$; ○, $p = 2.0$.

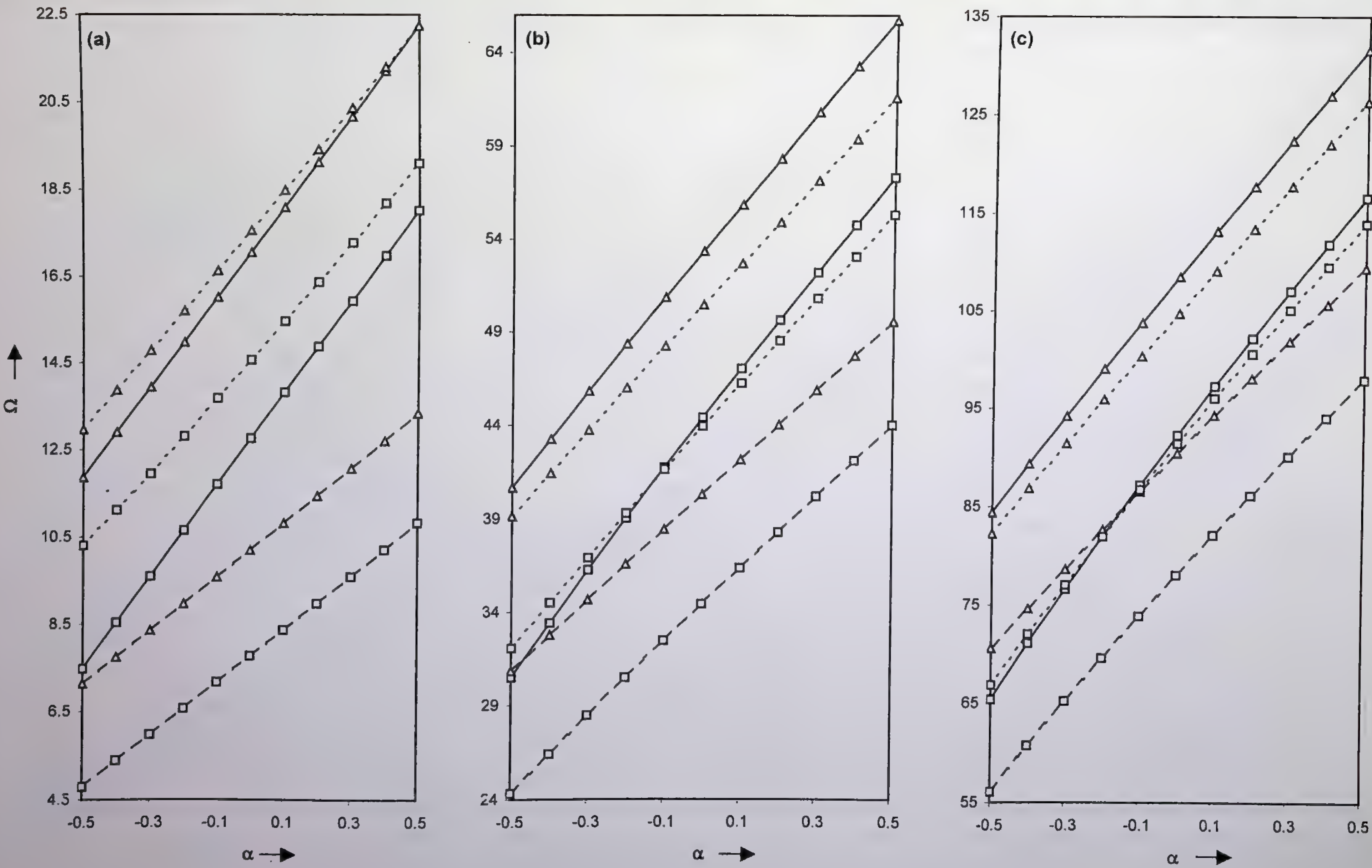

Fig. 9.3: Frequency parameter of plates vibrating in (a) fundamental (b) second and (c) third mode for $\zeta = 0.5$, $p = 5.0$.
———, clamped; – – – – –, simply supported; ----------, free. □, $\beta = 0.0$; Δ, $\beta = 0.5$.

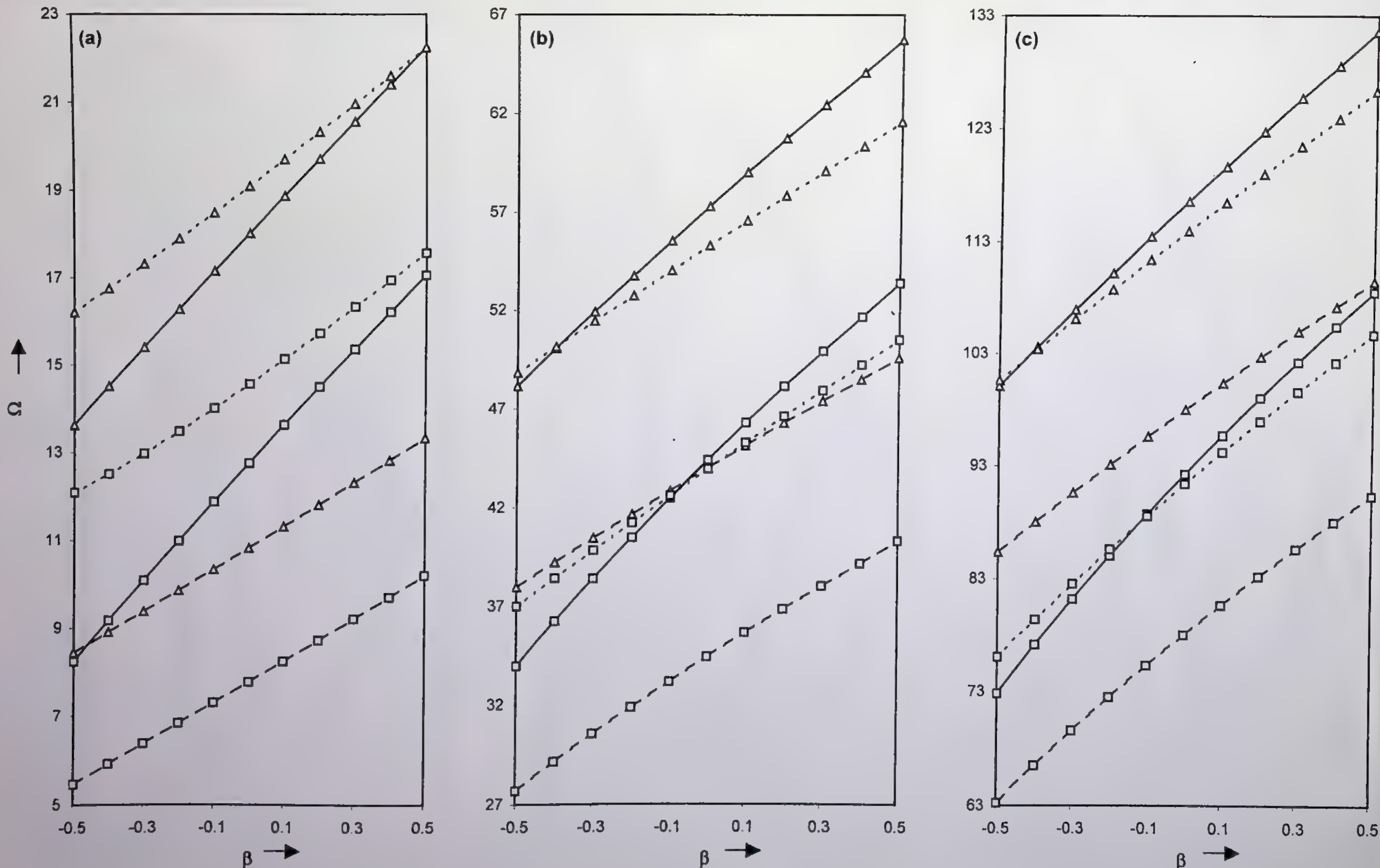

Fig. 9.4 : Frequency parameter of plates vibrating in (a) fundamental (b) second and (c) third mode for $\zeta = 0.5$, $p = 5.0$.
———, clamped; – – – – –, simply supported; -----------, free. □, $\alpha = 0.0$; Δ, $\alpha = 0.5$.

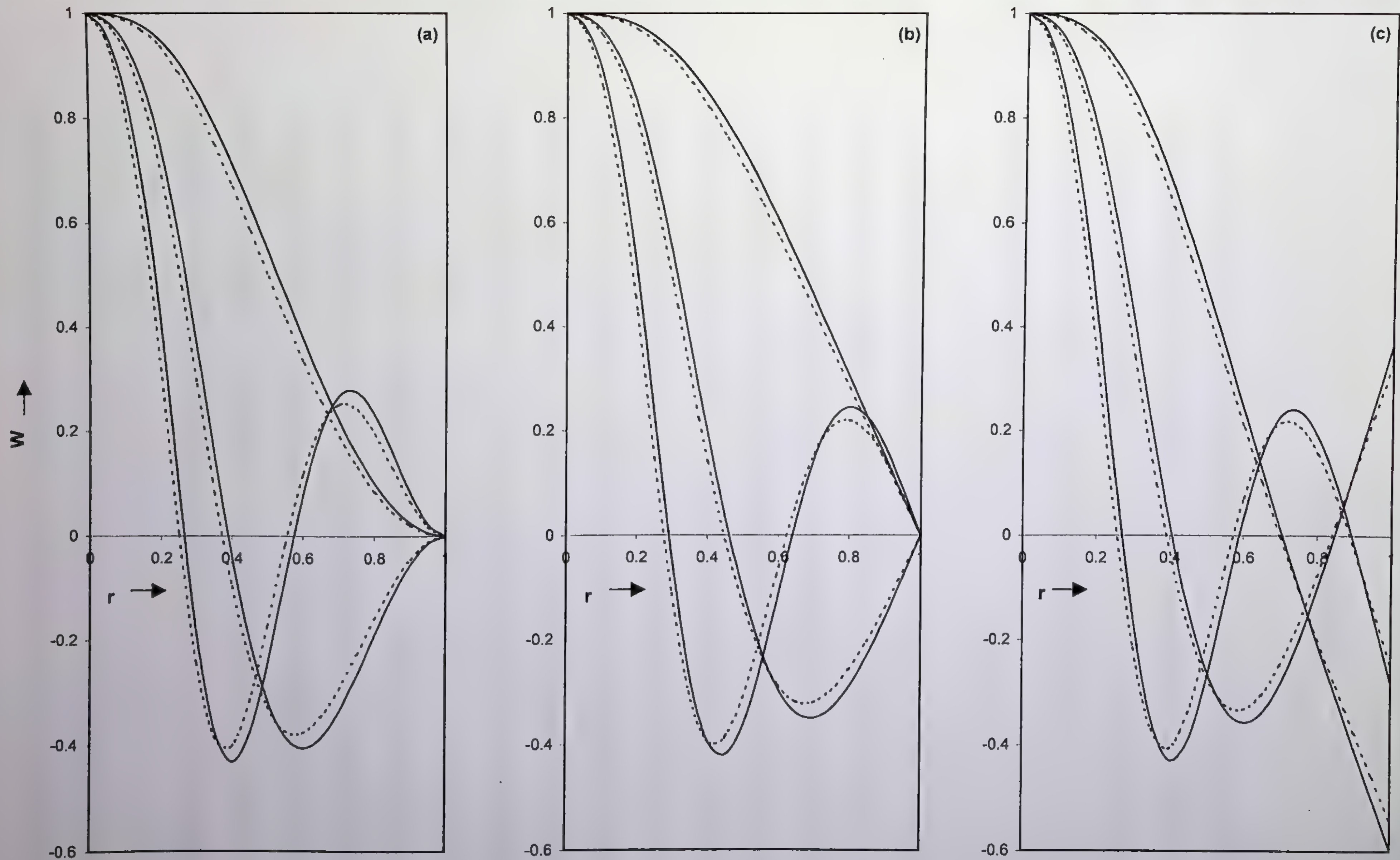

Fig. 9.5 : Normalized displacement for (a) clamped (b) simply supported and (c) free plates for $\alpha = 0.5$, $\beta = 0.5$, $p = 5.0$. ———, $\zeta = 0.0$; ----------, $\zeta = 0.5$.

CONCLUSION

The present work analyses free axisymmetric vibrations of isotropic and polar orthotropic non-homogeneous circular and annular plates of non-uniform thickness with various complicating effects such as elastic foundation, shear deformation, rotatory inertia, thermal gradient and elastically restrained edge. The analysis is of practical importance due to the use of such plates in various technological situations.

The governing differential equations of motion for various plate models considered in different chapters based on classical plate theory(CPT) and first order shear plate theory of Mindlin (SPT) have been derived using Hamilton's energy principle. Four different numerical methods namely Ritz method and differential quadrature method(DQM), new version of differential quadrature method(NDQM) and Chebyshev collocation technique have been employed to solve the above resulting equations with variable coefficients. Convergence and accuracy studies for all the methods have been carried out to reveal the convergence characteristics of particular method as well as to ensure the accuracy of the results. The present choice of collocation methods provides a faster convergence as compared to finite difference method, finite element method, quintic splines method and characteristic orthogonal polynomials method etc. In DQM and NDQM the collocation points have been chosen as zeros of shifted Chebyshev polynomials. This choice of collocation points has faster rate of convergence as compared to other types of collocation points i.e. equally spaced points and zeros of shifted Legendre polynomials. Further, NDQM converges faster than DQM while the convergence characteristics of DQM and Chebyshev collocation technique are more or less the same.

In case of circular plates, the frequency parameter Ω for free plate is smaller than that for clamped plate and greater than that for simply supported plate. The frequency parameter

increases with the increasing values of non-homogeneity parameter μ, taper parameters α and β, rigidity ratio p and flexibility parameter K_φ while it decreases with increase in density parameter η and thermal gradient ζ. This increase/decrease in frequency parameter further increases with the increasing order of modes. In case of plates for which thickness increases towards the outer edge, the frequency parameter for linear thickness variation ($\alpha > 0, \beta = 0$) is higher than that for parabolic thickness variation ($\alpha = 0, \beta > 0$) and smaller than that for quadratic thickness variation ($\alpha > 0, \beta > 0$). The behaviour is just the reverse when thickness decreases towards the outer edge.

In case of annular plates, the frequency parameter Ω for C-S plate is smaller than that for C-C plate and greater than that for C-F plate. The effect of various plate parameters such as μ, η and α, β on the frequency parameter remains almost similar to that of circular plates. For all the three boundary conditions, the frequency parameter is found to increase with the increase in values of foundation parameters K and G as well as the hole size of the plate, whatever be the values of other plate parameters. It is observed that the natural frequencies obtained by Pasternak foundation model are higher than that for Winkler model. The effect of foundation decreases with the increase in the number of modes.

In case of annular plates, the radii ratio plays an important role on the behaviour of frequency parameter Ω, which is found to increase with its increasing values. The values of radii ratio $\varepsilon(=b/a)$ from 0.1 to 0.7 have been taken from the literature. However, efforts were made to consider two extreme values of ε when it is equal to zero and when it approaches to unity. In case when ε is very small there is instability in the values of frequency parameter Ω. Further, for $\varepsilon = 0$ the annular plate reduces to circular plate and the two point boundary value problem becomes a boundary value problem. In case, when ε approaches unity, the plate reduces to a ring and the domain (ε, 1) becomes very small. The present collocation techniques give

converging results even for the half of the number of collocation points for $\varepsilon = 0.8$ and 0.9. The corresponding frequencies are sufficiently high which may be attributed to the reduced mass of the plate. However in this case the theory of rings should be applied instead of applying the theory of plates.

For moderately thick circular/annular plates, the frequency parameter Ω has been computed using first order shear plate theory of Mindlin (Ω_s) and classical plate theory (Ω_c). It is found that (Ω_c - Ω_s) increases with the increasing values of non-homogeneity parameter μ, taper parameters α and β, thickness parameter h_0 and with decreasing values of density parameter η. It also increases with the increase in order of modes. Thus the effects of rotatory inertia and transverse shear cannot be neglected while predicting the vibrational behaviour of non-homogeneous moderately thick plates ($h_0 > 0.1$). Similar inferences were drawn by Deresiewicz and Mindlin[1955], Gupta et al.[1994], Gupta and Lal[1985] in case of isotropic/orthotropic circular/annular plates.

Further, it is found that in case of CPT, the effect of non-homogeneity parameter μ ($\mu = 0.5$) for small value of density parameter η ($\eta = 0.1$) on the frequency parameter is found to increase by 15.7%, 11.3% and 9.2%, respectively for clamped, simply supported and free isotropic circular plates vibrating in fundamental mode. Similarly for $\mu = 0.1$ and $\eta = 0.5$, the frequency parameter is found to decrease by 5.7%. 8.2% and 11.4%, respectively. The above percentages are found to decrease and increase respectively with the increase in the number of modes. Moreover, the effect of non-homogeneity due to thermal gradient ζ ($\zeta = 0.3$) is to decrease the frequency parameter by 5.6%, 7.8% and 8.5% for clamped, simply supported and free isotropic circular plates, respectively. These percentage changes are found to decrease with the increase in orthotropy and also when the plate becomes thicker and thicker towards the outer edge. In case of isotropic annular plates, the effect of non-homogeneity parameter μ ($\mu = 0.5$) for $\eta = 0.1$

on the frequency parameter is found to increase by 14.3%, 12.3% and 8.3% for C-C, C-S and C-F boundary conditions, respectively, in the fundamental mode, while for $\mu = 0.1$ and $\eta = 0.5$, the frequency parameter decreases by 12.2%, 13.6% and 17.6%, respectively. These percentage changes decrease with the incorporation of foundation and increase with increase in hole size.

In case of SPT, the overestimation of fundamental frequency for non-homogeneous circular plate ($\mu = 0.5$, $\eta = 0.1$), in per cent by CPT with respect to SPT is 2.7%, 7.8% and 1.4% for clamped. simply supported and free edges, respectively for $h_0 = 0.1$. The corresponding percentage change in case of C-C, C-S and C-F annular plates are 15.2%, 9.8% and 2.2%, respectively. These percentage changes increase with the increase in the thickness parameter h_0 of the plate and with the increasing order of modes.

Scope for Future Work

All the problems considered in the thesis can be extended to asymmetric vibrations. Further, the analysis can be extended to study the natural frequencies of non-homogeneous plates of uniform and non-uniform thickness subjected to in-plane force due to their engineering applications.

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